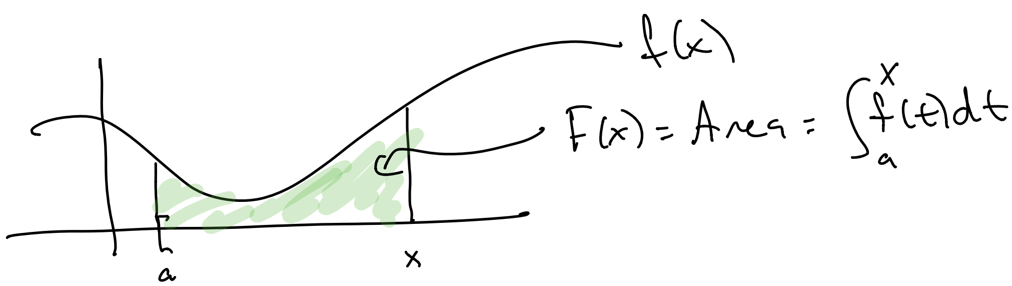
Fundamental Theorem of Calculus II

Recall: FTC I: If $f(x)$ is a function,

$$F(x) = \int_a^x f(t)\,dt \quad \text{then } F'(x) = f(x)$$

in other words, $F(x)$ is an anti-derivative for $f(x)$.



Lets evaluate $\int_1^5 (x^2+1)\,dx$.

Idea: define $F(x) = \int_1^x (t^2+1)\,dt$, then will

plug in $x=5 \rightsquigarrow F(5)$

1: Find $F(x)$

know $F(x)$ is an antiderivative for x^2+1

but we know how to write all antiderivatives:

$$\int (x^2+1)\,dx = \frac{1}{3}x^3 + x + C \quad \text{for some } C.$$

So, we can write $F(x) = \frac{1}{3}x^3 + x + C$ some C.

To find C: use the fact that

$$F(1) = \int_1^1 (x^2+1)\,dx = 0$$

$$0 = \frac{1}{3}(1)^3 + 1 + C \implies C = -\left(\frac{1}{3}(1)^3 + 1\right)$$

$$\implies F(x) = \frac{1}{3}x^3 + x - \left(\frac{1}{3}(1)^3 + 1\right)$$

2. Plug in: $F(5) = \left(\frac{1}{3}5^3 + 5\right) - \left(\frac{1}{3}(1)^3 + 1\right)$

$$\implies \int_1^5 (x^2+1)\,dx = \left(\frac{1}{3}5^3 + 5\right) - \left(\frac{1}{3}1^3 + 1\right)$$

$x=5$ $\quad$ $x=1$

what just happened?

found the anti-derivative $\frac{1}{3}x^3 + x$

plugged in 5 & 1 then subtracted.

FTC II

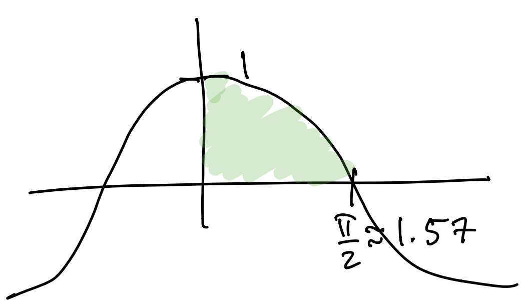
$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where $F(x)$ is <u>any</u> antiderivative for $f(x)$.

<u>examples</u>

$$\int_1^2 (x^2+3)\,dx = \left(\frac{1}{3}(2)^3 + 3(2)\right) - \left(\frac{1}{3}(1)^3 + 3(1)\right)$$

anti-derivative: $\frac{1}{3}x^3 + 3x$

$$\int_0^{\pi/2} \cos x\,dx = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$



$$\int_0^1 x^2\,dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Practice

$$\int_0^{\pi} \sin x\,dx = (-\cos \pi) - (-\cos 0) = (-(-1)) - (-1)$$
$$= 2$$

$$\int_1^4 \sqrt{x}\,dx = \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2}$$
$$= \frac{2}{3}8 - \frac{2}{3}1 = \frac{2}{3}(8-1) = \frac{14}{3}$$

Notational shorthand

$$F(b) - F(a) = \Big[F(x)\Big]_a^b = \Big[F(x)\Big]_{x=a}^{x=b}$$
$$= F(x)\Big]_a^b$$
$$= F(x)\Big|_a^b$$

FTC II: $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b$ if $F'(x) = f(x)$.

u-substitution

$$\int_1^5 (\sin e^x) e^x \, dx$$

Method 1: rewrite problem first to find antiderivative.

$$\int_1^5 (\sin e^x) e^x \, dx$$

$$\int (\sin e^x) e^x \, dx = \int \sin u \, du$$

$u = e^x$

$du = e^x dx$

$$= -\cos u + C$$

$$= -\cos e^x + C$$

Found that $-\cos e^x$ is an antiderivative.

$$\int_1^5 (\sin e^x) e^x \, dx = -\cos e^x \Big]_1^5$$

$$= (-\cos e^5) - (-\cos e^1)$$

Method 2: don't rewrite problem

$$\int_1^5 (\sin e^x) e^x \, dx = \int_{x=1}^{x=5} \sin u \, du$$

$u = e^x$

$du = e^x dx$

$$= -\cos u \Big]_{x=1}^{x=5}$$

$$= -\cos e^x \Big]_{x=1}^{x=5}$$

$$= \left(-\cos e^5\right) - \left(-\cos e^1\right)$$

closer look

~~Method~~ 2: don't rewrite problem

$$\int_1^5 (\sin e^x) e^x \, dx = \int_{\cancel{x=1}\ u=e^1}^{\cancel{x=5}\ u=e^5} \sin u \, du$$

$u = e^x$

$du = e^x dx$

$$= -\cos u \Big]_{x=1\ \ u=e^1}^{x=5\ \ u=e^5}$$

$$= -\cos e^x \Big]_{x=1}^{x=5}$$

$e^5 = u$ $x=5$

$$= \left(-\cos e^5\right) - \left(-\cos e^1\right)$$

$x=5$

$u=e^5$

Method 3 Convert entirely to u's.
(the good one)

$$\int_1^5 (\sin e^x) e^x \, dx = \int_{e^1}^{e^5} \sin u \, du = -\cos u \Big]_{e^1}^{e^5}$$

$u = e^x$ if $x=1$ then $u = e^1$

$du = e^x \, dx$ if $x=5$ then $u = e^5$

$$(-\cos e^5) - (-\cos e^1)$$

examples

$$\int_0^{\pi/2} \sin x \cos x \, dx = -\int_1^0 u \, du = -\left[\tfrac{1}{2}u^2\right]_1^0 = 0 - \left(-\tfrac{1}{2}1^2\right) = \tfrac{1}{2}$$

$u = \cos x$ if $x=0$ $u = \cos 0 = 1$

$du = -\sin x \, dx$ if $x = \pi/2$ $u = \cos\frac{\pi}{2} = 0$

$$\int_e^{e^e} \frac{1}{x \ln x} \, dx = \int_e^{e^e} \frac{1}{\ln x} \cdot \frac{1}{x} \, dx = \int_1^e \frac{1}{u} \, du$$

$u =$

$u = \ln x$ $x = e \Rightarrow u = \ln e = 1$

$du = \frac{1}{x} dx$ $x = e^e \Rightarrow u = \ln e^e = e$

$$\ln|u| \Big]_1^e = \ln e - \ln 1 = 1 - 0 = \boxed{1}$$