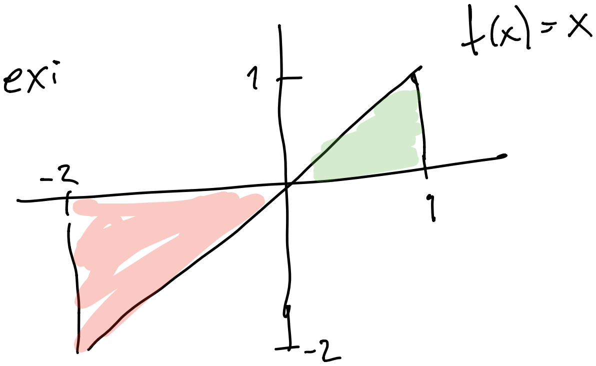
Definite Integrals (& the fundamental theorem of Calculus)

textbook 13.1/13.2 — textbook 14.1/14.2

Reminder Definite Integral = "signed" area between graph & x-axis

$$\int_a^b f(x)\,dx$$

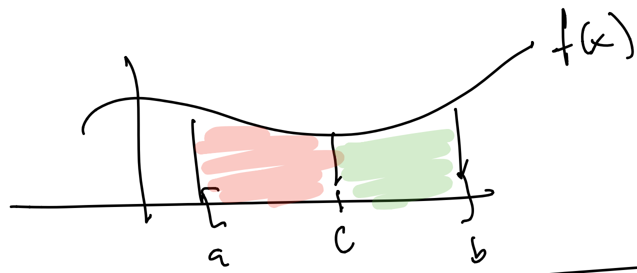
ex:



$$\int_0^1 x\,dx = \frac{1}{2} \qquad \int_{-2}^{0} x\,dx = -2$$

$$\int_{-2}^{1} x\,dx = (\text{area above}) - (\text{area below}) = \frac{1}{2} - 2 = -\frac{3}{2}$$

Properties of Definite Integrals



$$\boxed{\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx}$$

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx$$

$$\int_a^a f(x)\,dx + \int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

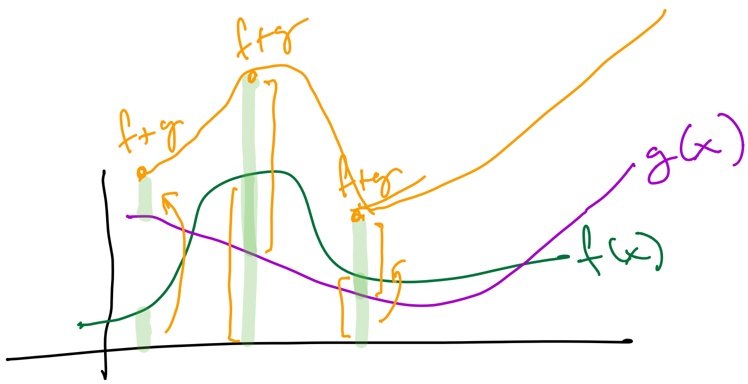
$$\Longrightarrow \int_a^a f(x)\,dx = 0 \Rightarrow$$

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = 0$$

$$\Rightarrow \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

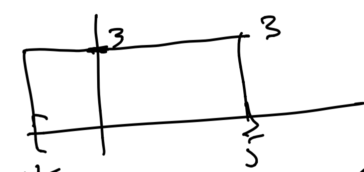
$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

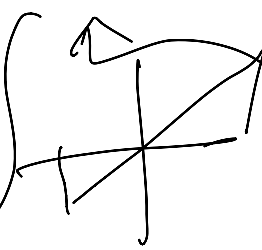
$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \quad \checkmark$$



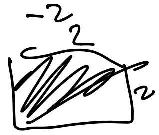
ex:

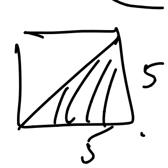
$$\int_{-2}^{5} (3-2x)\,dx = \underbrace{\int_{-2}^{5} 3\,dx}_{21} - 2\underbrace{\int_{-2}^{5} x\,dx}$$





$$\int_{-2}^{5} x\,dx = \int_{-2}^{0} x\,dx + \int_{0}^{5} x\,dx \quad \left(-\frac{4}{2} + \frac{25}{2}\right)$$

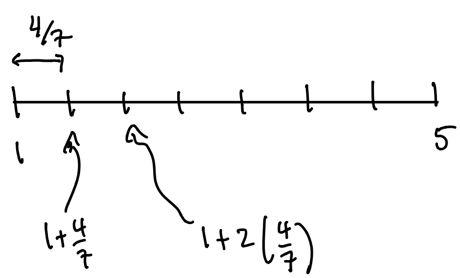




Riemann Sums (with summation notation)

example: Approximate $\int_1^5 e^x\,dx$ using 7 rectangles and right endpoints.

$a=1 \quad b=5 \quad n=7 \quad \Delta x = \frac{b-a}{n} = \frac{5-1}{7} = \frac{4}{7}$



segment 1: $[1, 1+4/7]$ $\quad x_1^* = 1+4/7$

segment 2: $[1+4/7, 1+2(4/7)]$ $\quad x_2^* = 1+2(4/7)$

(bases of rectangles)

choose x_i^* in ith segment.

$x_1^* = 1+4/7 \quad x_2^* = 1+2(4/7), \ \ldots \ x_i^* = 1+i(4/7)$

area of ith rectangle: $b \cdot h = \Delta x\, f(x_i^*)$

$$= (4/7)\, e^{1+i(4/7)}$$

total area:

$$(4/7)\,e^{1+1(4/7)} + (4/7)\,e^{1+2(4/7)} + (4/7)\,e^{1+3(4/7)} + \cdots + (4/7)\,e^{1+7(4/7)}$$

$$= \sum_{i=1}^{7} (4/7)\, e^{1+i(4/7)}$$

Practice:

Approximate $\int_{-1}^{2} (x^2+1)\,dx$ using left endpoints and 5 rectangles.

write in summation notation.

$$a=-1 \quad b=2 \quad n=5 \quad \Delta x = \frac{2-(-1)}{5} = \frac{3}{5}$$

$$x_1^* = -1 \qquad x_2^* = -1+\frac{3}{5} \quad x_3^* = -1+2\left(\frac{3}{5}\right)$$

$$x_i^* = -1+(i-1)\frac{3}{5}$$

$$\text{Area} = \Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \Delta x\, f(x_3^*) + \dots$$

$$= \frac{3}{5}\left(1+(-1)^2\right) + \frac{3}{5}\left(1+\left(-1+\frac{3}{5}\right)^2\right) + \frac{3}{5}\left(1+\left(-1+2\left(\frac{3}{5}\right)\right)^2\right) + \dots$$

$$= \sum_{i=1}^{n} \Delta x\, f(x_i^*) = \sum_{i=1}^{n} \left(\frac{3}{5}\right)\left(1+\left(-1+(i-1)\left(\frac{3}{5}\right)\right)^2\right)$$

$$\sum_{i=0}^{4} \left(\frac{3}{5}\right)\left(1+\left(-1+i\left(\frac{3}{5}\right)\right)^2\right)$$

ex: $$\sum_{i=1}^{n} (a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c b_i = c \sum_{i=1}^{n} b_i$$

$$cb_1 + cb_2 + \dots + cb_n$$
$$= c(b_1+b_2+\dots+b_n)$$

$$\sum_{i=1}^{n} 1 = \underbrace{1+1+\cdots+1}_{n \text{ times}} = n.$$

$$\sum_{i=0}^{4}\left(\frac{3}{5}\right)\left(1+\left(-1+i\left(\frac{3}{5}\right)\right)^2\right) = \frac{3}{5}\sum_{i=0}^{4} 1+\left(-1+i\left(\frac{3}{5}\right)\right)^2$$

$$= \frac{3}{5}\left[\sum_{i=0}^{4} 1 + \sum_{i=0}^{4}\left(-1+i\left(\frac{3}{5}\right)\right)^2\right]$$

$$= \frac{3}{5}\left[5 + \sum_{i=0}^{4}\left(1 - 2\cdot 1\cdot i\left(\frac{3}{5}\right) + i^2\left(\frac{3}{5}\right)^2\right)\right]$$

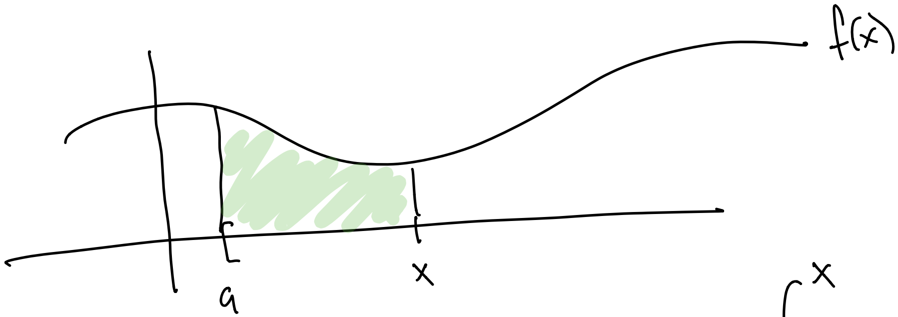
$$= \frac{3}{5}\left[5 + \sum_{i=0}^{4} 1 - 2\cdot\left(\frac{3}{5}\right)\sum_{i=0}^{4} i + \left(\frac{3}{5}\right)^2\sum_{i=0}^{4} i^2\right]$$

$$= \frac{3}{5}\left[5+5-\frac{6}{5}(1+2+3+4)+\frac{9}{25}\left(1^2+2^2+3^2+4^2\right)\right]$$

$$= \frac{3}{5}\left[10-\frac{6}{5}(10)+\frac{9}{25}(30)\right]$$

Fundamental Theorem of Calculus (FTC I)

Question: What is the rate of change of area under a graph?



$$F(x) = \int_a^x f(t)\,dt$$

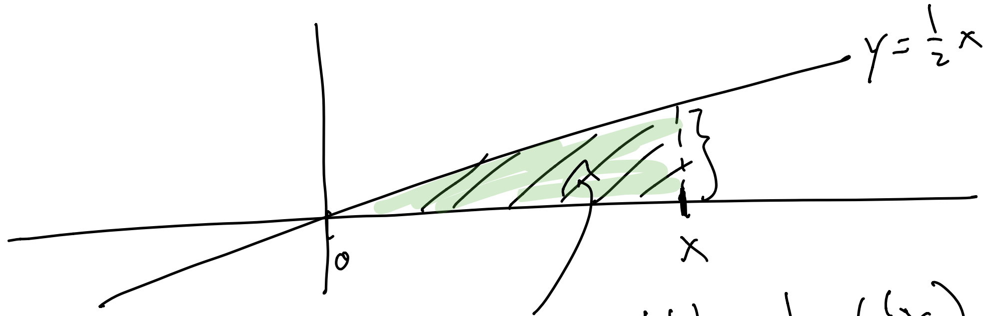
Intuitive FTC I: rate of change of area under curve from a to x is proportional to size of f(x).

Actual Thm (FTC I)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

ex: $f(x) = \frac{1}{2}x$ $\quad F(x) = \int_0^x f(x)\,dx = \frac{1}{4}x^2$

$$F'(x) = 2\cdot\frac{1}{4}x^1 = \frac{1}{2}x = f(x)$$



$$\text{area} = \frac{1}{2}bh = \frac{1}{2}x\left(\frac{1}{2}x\right)$$

$$= \frac{1}{4}x^2$$