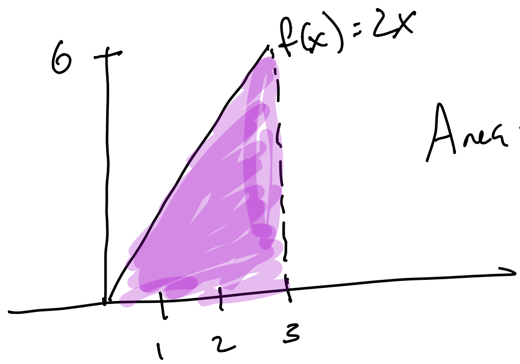
Definite Integrals & Riemann Sums

textbook 13.1 / 13.2

Definite integral = "signed" area between graph & x-axis.

ex: definite integral of $f(x) = 2x$ between $x=0$ & $x=3$

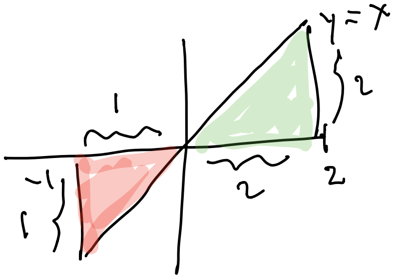


Area $= \frac{1}{2} b \cdot h = \frac{1}{2}(18) = 9$

Notation: $\displaystyle\int_0^3 2x\,dx = 9$

$\left(\text{Compare } \displaystyle\int 2x\,dx = x^2 + C \right)$

signed area = area below x-axis counts as negative.

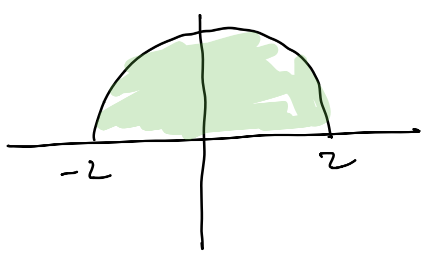


$$\int_{-1}^{2} x\,dx = \left(\frac{1}{2}\,2\cdot 2\right) - \left(\frac{1}{2}\,1\cdot 1\right)$$

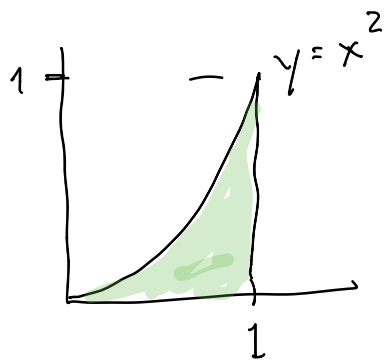
How do we compute these?

Lucky?

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi 2^2$$

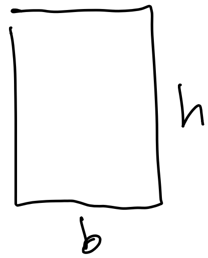


$$y = \sqrt{4-x^2}$$
$$y^2 = 4-x^2$$
$$x^2+y^2 = 2^2$$



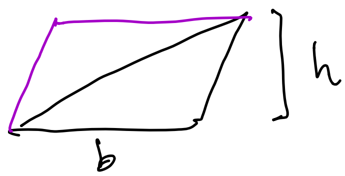
$$\int_{0}^{1} x^2\,dx = ?$$ How?

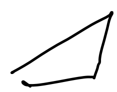
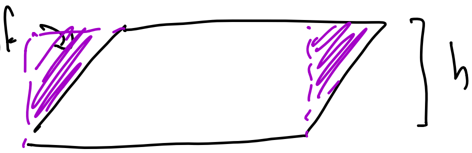
Most basic area: Rectangle.



$A = bh$

because.

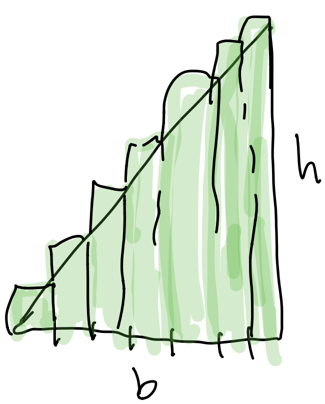


 $= \frac{1}{2}$ of 

$A = \frac{1}{2} bh$

parallelogram

Great Idea: Riemann said "use lots of rectangles"



each rectangle has a base & ht.

total area = sum of all areas of rectangles (overestimate)

bases are all $\frac{1}{7} b = \frac{b}{7}$

heights?

tallest = h

each one $\frac{1}{7} h$ bigger

$h_1 = \frac{1}{7} h \quad h_2 = \frac{2}{7} h, \ldots, h_7 = \frac{7}{7} h$

total area $= b_1h_1 + b_2h_2 + \cdots + b_7h_7$

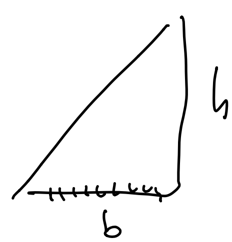
$$\left(\frac{b}{7}\right)\left(\frac{1\cdot h}{7}\right) + \left(\frac{b}{7}\right)\left(\frac{2h}{7}\right) + \left(\frac{b}{7}\right)\left(\frac{3h}{7}\right) + \left(\frac{b}{7}\right)\left(\frac{4h}{7}\right) + \left(\frac{b}{7}\right)\left(\frac{5h}{7}\right) +$$
$$\left(\frac{b}{7}\right)\left(\frac{6h}{7}\right) + \left(\frac{b}{7}\right)\left(\frac{7h}{7}\right)$$

$$\left(\frac{b}{7}\right)\left(\frac{h}{7}\right)(1+2+3+4+5+6+7) = \left(\frac{b}{7}\right)\left(\frac{h}{7}\right)(8)\left(\frac{7}{2}\right)$$

$$= \frac{8\cdot 7}{7\cdot 7\cdot 2}bh = \frac{8\cdot 7}{7\cdot 7}\frac{1}{2}bh$$

$$= \frac{8}{7}\frac{1}{2}bh$$

n rectagles, contiuing pattrn.



$A = b_1h_1 + b_2h_2 + \cdots + b_nh_n$

$$b_i = \frac{b}{n}$$

$$h_n = h$$

$$h_1 = \frac{1}{n}h$$
$$h_2 = \frac{2}{n}h$$
$$\vdots$$
$$h_n = \frac{n}{n}h = h$$

$$A = \left(\frac{b}{n}\right)\left(\frac{1}{n}h\right) + \left(\frac{b}{n}\right)\left(\frac{2}{n}h\right) + \cdots + \left(\frac{b}{n}\right)\left(\frac{n}{n}h\right)$$

$$A = \left(\frac{b}{n}\right)\left(\frac{h}{n}\right)(1+2+3+\cdots+n)$$

$$= \left(\frac{b}{n}\right)\left(\frac{h}{n}\right)\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)}{n^2}\,\frac{1}{2}bh$$

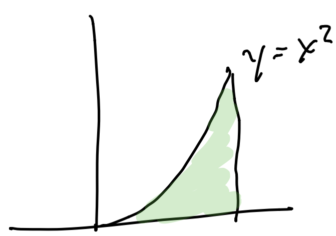
$$= \frac{n^2+n}{n^2}\,\frac{1}{2}bh$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{1}{2}bh\right)$$

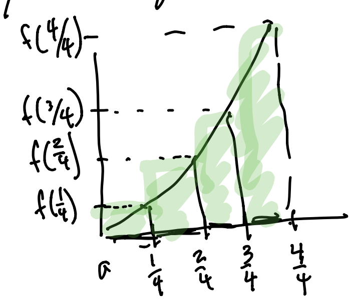
$$\lim_{n\to\infty}\left(1+\overset{\to 0}{\frac{1}{n}}\right)\frac{1}{2}bh = (1+0)\,\frac{1}{2}bh = \boxed{\frac{1}{2}bh}$$

Area under $y = x^2$ from 0 to 1

$$\int_0^1 x^2\,dx$$



Strategy: rectangles.



$$A = b_1h_1 + b_2h_2 + b_3h_3 + b_4h_4$$

$$b_i = \frac{1}{4}$$

$$h_1 = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2$$

$$h_2 = f\left(\frac{2}{4}\right) = \left(\frac{2}{4}\right)^2$$

$$\vdots$$

$$A = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)\left(\frac{2}{4}\right)^2 + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)\left(\frac{4}{4}\right)^2$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^2 \left(1^2 + 2^2 + 3^2 + 4^2\right)$$

$$= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^2 \left(1 + 4 + 9 + 16\right) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^2 (30) = \frac{30}{4^3}$$

$$= \frac{30}{64}$$

=

<u>with n rectangles:</u> bases $= \frac{1}{n}$

$$h_1 = \left(\frac{1}{n}\right)^2 \quad h_2 = \left(\frac{2}{n}\right)^2 \quad \ldots \quad h_n = \left(\frac{n}{n}\right)^2$$

$$= f\left(\frac{1}{n}\right) \qquad = f\left(\frac{2}{n}\right) \qquad = f\left(\frac{n}{n}\right)$$

$$A = b_1h_1 + b_2h_2 + \ldots + b_nh_n$$

$$= \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$= \left(\frac{1}{n^3}\right)\left(1^2 + 2^2 + \ldots + n^2\right) = \left(\frac{1}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{n(n+1)(2n+1)}{n^3 \cdot 6}$$

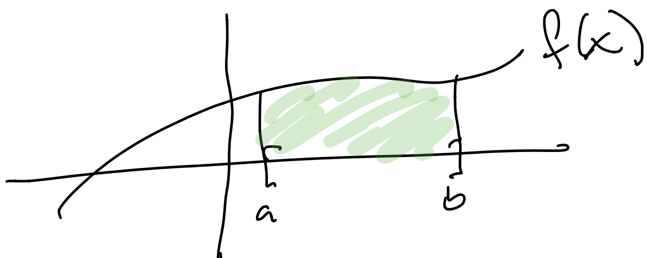
$$= \frac{n(2n^2 + 2n + n + 1)}{6n^3} = \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$\text{Area} = \frac{2n^3+3n^2+n}{6n^3} = \frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}$$

$$= \frac{1}{3} + \frac{1}{2}\frac{1}{n} + \frac{1}{6}\frac{1}{n^2}$$

if $n \to \infty$, this area $\to \boxed{\frac{1}{3}}$

To find (or approximate) definite integrals using Riemann sums, procedure is:

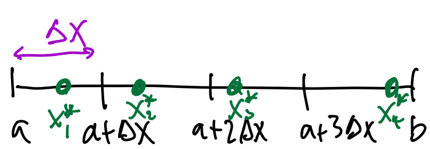


$$\int_a^b f(x)\,dx$$

divide up interval from a to b into n equally sized intervals (length of each is $\frac{b-a}{n}$ = "Δx")

ht of rectangles determined by some point on graph in the interval.

i.e. in the i^{th} interval, choose some point x_i^*, then ht of i^{th} interval is $f(x_i^*)$



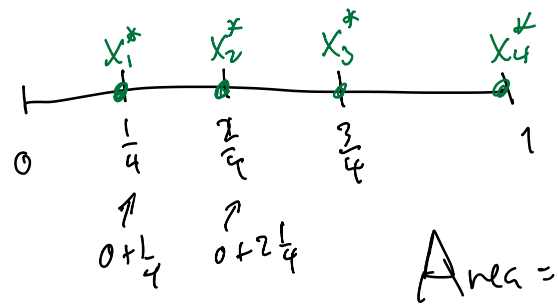
Area of rectangles $= (\Delta x) f(x_1^*) + (\Delta x) f(x_2^*) + \cdots + (\Delta x) f(x_n^*)$

<u>Theorem (Riemann)</u>

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left((\Delta x) f(x_1^*) + (\Delta x) f(x_2^*) + \cdots + (\Delta x) f(x_n^*)\right)$$

<u>ex:</u> 4 rectangles for $f(x) = x^2$ $\quad a=0 \quad b=1$

approximate $\int_0^1 x^2\,dx$ using right endpoints



$\Delta x = \frac{1}{4}$

Area $= (\Delta x) f(x_1^*) + \Delta x\, f(x_2^*) \cdots$

$= \left(\frac{1}{4}\right) f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \cdots$

$= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^2 + \cdots$