u-substitution (anti-chain rule)

$$\int 2x\sin(x^2)\,dx = \int \underbrace{\sin(x^2)}_{\sin u}\ \underbrace{2x\,dx}_{du} = \int \sin u\,du$$

$u = x^2$

$du = u'(x)\,dx$

$= 2x\,dx$

$= -\cos u + C$

$= -\cos(x^2) + C$

$$\int \sin(u(x))\,\frac{du}{dx}\,dx$$

$$\int \frac{2x}{x^2-1}\,dx = \int \frac{1}{x^2-1}\,2x\,dx = \int \frac{1}{u}\,du$$

$u = x^2-1$

$du = 2x\,dx$

$= \ln|u| + C$

$= \ln|x^2-1| + C$

$$\int e^{x^2}x\,dx \longrightarrow \frac{1}{2}\int e^{x^2}2x\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int e^u \tfrac{1}{2}\,du = \frac{1}{2}\int e^u\,du$$

$= \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2} + C$

Practice:

1. $\int x^2 \sin x^3 \, dx$

$u = x^3$

2. $\int \frac{3x^2}{1+x^3} \, dx$

$u = 1 + x^3$

3. $\int \sin x \cos x \, dx = \int u(-1)\, du$

$\boxed{u = \cos x}$ or $\sin x$

$= -\int u \, du$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\frac{1}{2}u^2 + C$

$= -\frac{1}{2}\cos^2 x + C$

(other way: $\frac{1}{2}\sin^2 x + C$)

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx = \int \sin u \left(\frac{1}{\sqrt{x}}\right) dx$$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2} dx$

$2\,du = x^{-1/2} dx = \frac{1}{\sqrt{x}} dx$

$= \int \sin u \,(2)\, du$

$= 2\int \sin u \, du$

$= 2(-\cos u) + C$

$= -2\cos\sqrt{x} + C$

$$\int \frac{2x}{\sqrt{x^2-3}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$$

$u = x^2 - 3$

$du = 2x\,dx$

$$= \frac{1}{-\frac{1}{2}+1} u^{-\frac{1}{2}+1} + C$$

$$= 2u^{1/2} + C$$

$$= 2(x^2-3)^{1/2} + C$$

$$\int \frac{x^2+3}{x+1}\,dx = \int \frac{(u-1)^2+3}{u}\,du$$

$u = x+1$

$du = dx$

$u-1 = x$

$$\int \frac{x^2+3}{x}\,dx = \int (x^2+3)x^{-1}\,dx$$

$$= \int (x + 3x^{-1})\,dx$$