# Antiderivatives

## Practice

If we know $f'(x) = 4x$ what could $f(x)$ be?

$f(x) = 2x^2$

"power rule reverse engineering + modification of guess to make coeff. work"

$f'(x) = f(x)$ $e^x$ "pattern recognition"

$f'(x) = \sin x$ $f(x) = -\cos x$ "I've seen that before"

$f'(x) = \frac{1}{x}$ $f(x) = \ln x$

$f'(x) = \sec x$ $\tan x$? $\frac{d}{dx} \tan x = \sec^2 x$

$f(x) = ?$ "I don't know" $\sqrt{\tan x}$? $\frac{d}{dx} \sqrt{\tan x} = \frac{1}{2}(\tan x)^{-1/2} \cdot \sec^2 x$

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$f'(x) = f(x)^2$ $\frac{1}{3}x^3 \longrightarrow x^2$

$f$ $f' \overset{?}{=} f^2$

$f = x$ $f' = 1$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ $\left(-\frac{1}{x}\right)$ $f' = \frac{1}{x^2} = f^2$

$f$

Basic important type of problem:

differential equations (2700.. )

equations about functions & their derivatives.

Most fundamental type:

$\frac{dy}{dx}$ = function of x $\rightsquigarrow$ find y.

Antiderivatives

Def: We say that $F(x)$ is an antiderivative for $f(x)$ if $F'(x) = f(x)$.

First Case:

Anti-derivatives of $f(x) = 0$.

FACT: $F(x)$ is an antiderivative of 0 exactly when $F(x)$ = constant.

Suppose $F(x)$ & $G(x)$ are both anti-derivatives of $f(x)$.

then $F(x) - G(x)$ has derivative

$$\frac{d}{dx}\left(F(x) - G(x)\right) = F'(x) - G'(x) = f(x) - f(x) = 0$$

so $F(x) - G(x)$ is a constant.

$\Rightarrow$ any two anti-derivatives differ by a constant.

To find all anti-derivatives, find one, the rest differ by constants:

$f(x) = 2x$ find all anti-derivatives of $f(x)$

$F(x) = x^2$ , $x^2 - 1$ , $x^2 + 5$ , $x^2 + 100$ , ...

will write: $x^2 + C$ , $C$ any constant.

" $x^2 + C$ is the general form of an antiderivative for $f(x)$ "

Notation:

$\int f(x)dx$ means the general form of an antiderivative for $f(x)$

also called " the indefinite integral "

example:

$\int 2x\,dx = x^2 + C$

$\int \frac{dy}{dx}\,dx = y + C$

$\int \sin x\,dx = -\cos x + C$

$\int \sec^2 x\,dx = \tan x + C$

( in Physics: $\int dx\, 2x = x^2 + C$ )

Roles:

$$\int k f(x)\, dx = k \int f(x)\, dx$$

to find an antiderivatve for $k \cdot f(x)$, find an antiderivatve for $f(x)$ & multiply it by $k$.

ex

$$\int 2x\, dx = x^2 + C$$

$$\int 50x\, dx = 25 \int 2x\, dx = 25 \cdot x^2 + C$$

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Anti-Power role:

$$(x^n)' = n x^{n-1}$$

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

$$\int x^{-1}\, dx = \ln|x| + C$$

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1) $\int \left(x^3 + 3x^2 + 2 - \frac{1}{x^3}\right) dx$

$$= \int x^3\, dx + 3\int x^2\, dx + 2\int 1\, dx - \int x^{-3}\, dx$$

$$= \frac{1}{4} x^4 + 3 \frac{1}{3} x^3 + 2x - \left(\frac{1}{-2}\right) x^{-2} + C$$

2) $\int (3\cos x - \sin x)\, dx = 3\int \cos x\, dx - \int \sin x\, dx$

$= 3\sin x - (-\cos x) + C$

3) $\int (e^x - \sec x \tan x)\, dx = \int e^x dx - \int \sec x \tan x\, dx$

$= e^x - \sec x + C$

4) $\int e^{2x}\, dx = \frac{1}{2} e^{2x} + C$

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The anti-rules

anti-chain rule:

$F'(x) = f(x)$

$\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)$

$= f(g(x)) \cdot g'(x)$

$\int f(g(x)) g'(x)\, dx$

$= F(g(x)) + C$

usual notation: let $u = g(x)$

$\int f(u)\, u'(x)\, dx = F(u) + C$

"u-substitution"

$\int e^{\sin x} \cos x\, dx = e^u + C = e^{\sin x} + C$

$u = \sin x \quad u'(x) = \cos x$

$e^u$

$$\int (2x^2-5)^{10} \cdot 4x\,dx = \int u^{10}\,du = \frac{1}{11}u^{11} + C$$

$f(u) = u^{10}$ $\quad u = 2x^2-5$ $\quad = \frac{1}{11}(2x^2-5)^{11} + C$

$u' = 4x$

$$\int \sin(x^2)\,\underbrace{2x\,dx} = \int \sin u\,du = -\cos u + C$$

$$= -\cos x^2 + C$$

$u = x^2$

$du = u'(x)\,dx = 2x\,dx$

Anti-product rule

$$(fg)' = f'g + fg'$$

$$fg = \int f'g\,dx + \int fg'\,dx$$

In practice: $\int fg'\,dx = fg - \int f'g\,dx$

"Integration by parts"

$$\int x\sin x\,dx = -x\cos x - \int 1\cdot(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$f = x$ $\quad f' = 1$

$g' = \sin x$ $\quad g = -\cos x$ $\quad = -x\cos x + \sin x + C$