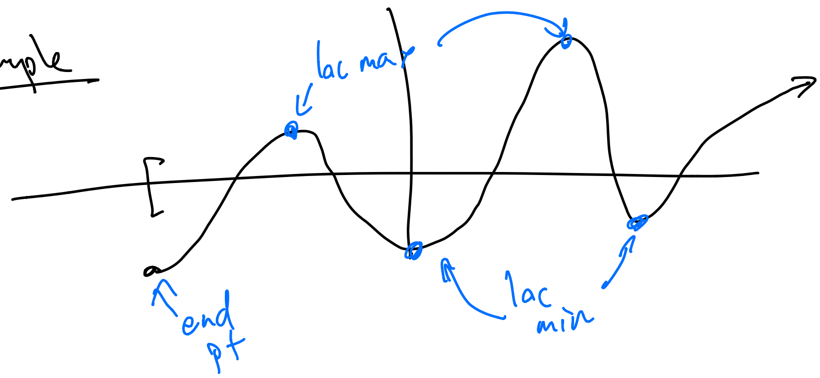
Infinite limits, horizontal asymptotes and L'hopital's rule

__Basic Questions__ Given a function $f(x)$ could ask: how do we make values large/small, what does the function do as x gets large (pos/neg)?

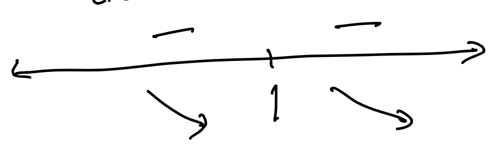
__Example__



? want $\lim\limits_{x \to \infty} f(x)$

$f(x) = \dfrac{x+1}{x-1}$ where does this get big?

From last time: $f'(x) = \dfrac{-2}{(x-1)^2}$



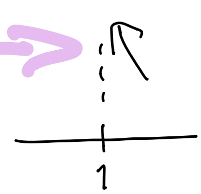
would next want to check what $f(x)$ does close to $1^{\pm}$, ∞, $-\infty$

$\lim\limits_{x \to 1^+} f(x)$ $\lim\limits_{x \to 1^-} f(x)$ $\lim\limits_{x \to \infty} f(x)$

$\lim\limits_{x \to -\infty} f(x)$

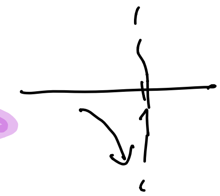
$\lim\limits_{x \to 1^+} \dfrac{x+1}{x-1} \approx \dfrac{2}{\text{small (pos)}} = \text{BIG (pos)}$

$= \infty$



$$\lim_{x\to 1^-} \frac{x+1}{x-1} \approx \frac{2}{\text{small(neg)}} = \text{BIG(neg)}$$

$$= -\infty$$



<u>Def</u> $\lim_{x\to\infty} f(x) = L$ means we can make $f(x)$ as close as we want to L by making x sufficiently large & positive.

<u>Def</u> $\lim_{x\to-\infty} f(x) = L$ means we can make $f(x)$ as close as we want to L by making x sufficiently large & negative.

<u>Basic Fact:</u> $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$

<u>Basic Strategy:</u> multiply through by lots of $1/x$'s to use FACT above.

$$\lim_{x\to\infty} \frac{x+1}{x-1} = \lim_{x\to\infty} \frac{(x+1)}{(x-1)} \frac{(1/x)}{(1/x)} = \lim_{x\to\infty} \frac{1+1/x}{1-1/x}$$

$$= \frac{\lim_{x\to\infty} 1 + \overbrace{\lim_{x\to\infty} 1/x}^{\text{Fact}\Rightarrow 0}}{\lim_{x\to\infty} 1 - \underbrace{\lim_{x\to\infty} 1/x}_{\text{Fact}\Rightarrow 0}} = \frac{1+0}{1-0} = 1$$

(same for $-\infty$)

words of caution

If you find yourself writing $\frac{\infty}{\infty}$ you generally did something wrong.

"indeterminate form"

meaning "try something else."

$$\lim_{x\to\infty} \frac{x+1}{x-1} = \frac{\infty}{\infty} = 1$$

$$\lim_{x\to\infty} \frac{x}{x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} 1 = 1$$

$$\lim_{x\to\infty} \frac{x}{x^2} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{x^2}{x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} x = \infty$$

$$\lim_{x\to\infty} \frac{x^2-2x+1}{3x^2-4} = \lim_{x\to\infty} \frac{(x^2-2x+1)1/x}{(3x^2-4)1/x}$$

$$= \lim_{x\to\infty} \frac{x-2+1/x}{3x-4/x}$$

good

bad

instead: $\lim_{x\to\infty} \frac{(x^2-2x+1)}{(3x^2-4)} \frac{1/x^2}{1/x^2}$

$$= \lim_{x\to\infty} \frac{1-2\,1/x+1/x^2}{3-4\,1/x^2}$$

$$= \frac{\lim_{x\to\infty} 1 - 2\lim_{x\to\infty} 1/x + \left(\lim_{x\to\infty} 1/x\right)\left(\lim_{x\to\infty} 1/x\right)}{3-4\left(\lim_{x\to\infty} 1/x\right)\left(\lim_{x\to\infty} 1/x\right)}$$

$\boxed{= \frac{1}{3}}$

Practice

1) $\lim\limits_{x\to\infty} \frac{3x}{x^2+4}$

$= \lim\limits_{x\to\infty} \frac{(3x)\,1/x^2}{(x^2+4)\,1/x^2}$

$= \lim\limits_{x\to\infty} \frac{3 \cdot \cancel{1/x}\to 0}{1 + 4 \cdot \cancel{1/x^2}\to 0} = \frac{0}{1} = 0$

2) $\lim\limits_{x\to\infty} \frac{\sin x}{x}$

(squeeze)

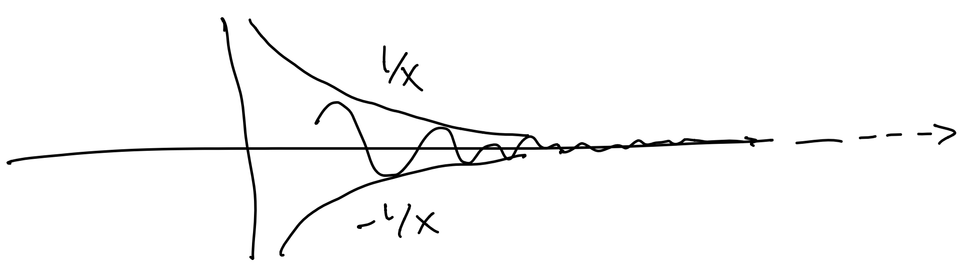
$-1 \le \sin x \le 1$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$x \to \infty$
so x is positive

$\lim\limits_{x\to\infty} -\frac{1}{x} = -\lim\limits_{x\to\infty} \frac{1}{x} = -0 = 0 = \lim\limits_{x\to\infty} \frac{1}{x}$

$\Rightarrow \lim\limits_{x\to\infty} \frac{\sin x}{x} = 0.$



$$\lim_{x\to\infty} \frac{x}{e^x}$$

Use L'Hopital's rule (due mostly to Bernoulli)

Rule says: If you want to figure out

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$ and if $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$ both either 0 or ∞

$a = \pm\infty$ is ok

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

example $$\lim_{x\to\infty} \frac{x}{e^x} \underbrace{= \frac{\infty}{\infty}}_{\text{"metaphor"}}$$

$\lim_{x\to\infty} x = \infty$

$\lim_{x\to\infty} e^x = \infty$ } can use L'hopital

$$= \lim_{x\to\infty} \frac{x'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{BIG} = \text{small}$$

$$= 0$$

ex: $$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$$ can use L'Hop.

$$= \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

Practice $$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$ $$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to\infty} \sqrt{x} = \infty$ } L'hop

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{(\ln x)'}{(\sqrt{x})'} = \lim_{x\to\infty}\frac{(1/x)}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} 2\,\frac{x^{1/2}}{x}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{x}}$$

$$= 2\sqrt{\underbrace{\lim_{x\to\infty}\frac{1}{x}}_{0}} = 0$$

$$\lim_{x\to\infty}\frac{e^x}{x^2}$$

$$\left.\begin{array}{l}\lim_{x\to\infty} e^x = \infty \\ \lim_{x\to\infty} x^2 = \infty\end{array}\right\} \text{L'Hop}$$

$$\lim_{x\to\infty}\frac{e^x}{2x} \quad \left\{\begin{array}{l}\lim_{x\to\infty} e^x = \infty \\ \lim_{x\to\infty} 2x = \infty\end{array}\right.$$

L'hop's

$$= \lim_{x\to\infty}\frac{e^x}{2} = \infty$$

If I'm thinking of a # between 1 & n & n people try to guess it, what's the prob. than none guess my number?

chance that an individual guesses: $\frac{1}{n}$

" " " doesn't: $1-\frac{1}{n}$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = L$$

take ln of both sides

$$\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^n = \ln L$$

$$\lim_{n\to\infty} n\ln\left(1-\frac{1}{n}\right) = \ln L$$

$$= \lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{n}\right)}{1/n}$$

$\frac{0}{0}$ so l'hop $\quad n\to\infty \quad \frac{1}{n}\to 0 \quad 1-\frac{1}{n}\to 1$

$\ln\left(1-\frac{1}{n}\right)\to \ln 1 = 0$

bottom $\to 0$

$$= \lim_{n\to\infty} \frac{\frac{1}{\left(1-\frac{1}{n}\right)}\cdot\left(+\frac{1}{n^2}\right)}{\left(-1/n^2\right)} = \lim_{n\to\infty} \frac{-1}{1-\frac{1}{n}} = -1$$

$$\ln L = -1 \qquad e^{-1} = L$$

$\frac{1}{e}$ = prob that no one guesses #