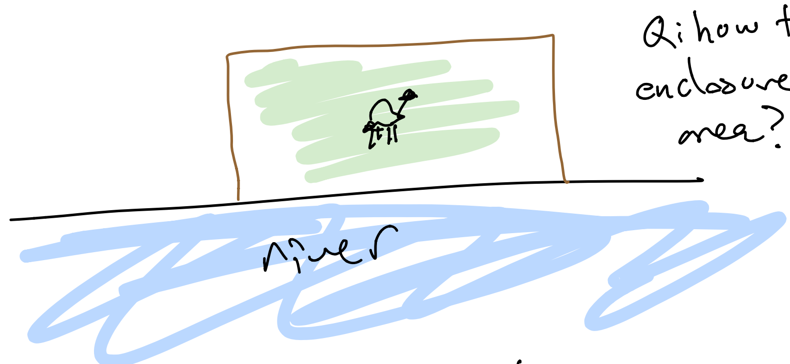
Famous farmer & fence problem

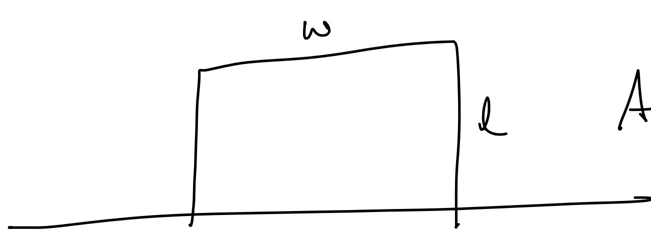


Have 100 ft of fence
Q: how to build rectangular enclosure w/ maximum area?

More precisely, what are the dimensions of rectangle w/ maximum area such that perimeter (along 3 sides) is exactly 100?

Strategy: Write Area as a function of something, then do what we need to do

take derivative, make sign chart ⇝ sketch ⇝ ?



$A = lw$

$2l + w = 100$

$w = 100 - 2l$

$A(l) = l(100 - 2l)$
$= 100l - 2l^2$

$A'(\ell) = 100 - 4\ell$ crit pt @ $\ell = 25$

$A'(\ell) = 0$

so clear that maximum value occurs at $\ell = 25$

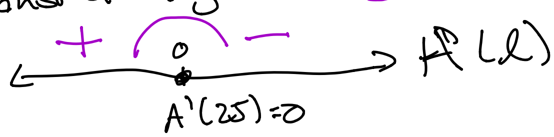
$A''(\ell) = -4 \Rightarrow$ function is always concave down.

thought experiment: pretend that we only knew

$A'(25) = 0$, $A''(\ell)$ negative always.

slopes always becoming more neg.

can reconstruct sign chart



Summary:

2nd derivative test: if $f'(a) = 0$ and $f''(a) > 0$ then a is a local min

- if $f'(a) = 0$ and $f''(a) < 0$ then a is a local max

Answer for fence

$\ell = 25$

$w = 50$

Variation Suppose the farmer is legally obligated to use all fencing, but because of a bad insurance policy, doesn't want to keep any animals.

Q: how can farmer minimize area & still use all fencing?

Need to use: DOMAIN!

$A(\ell) = \ell(100-2\ell)$

$= \ell w$

$w = 100-2\ell$

$0 \le \ell \le 100$?

$0 \le \ell$ $\quad$ $0 \le w = 100-2\ell$

$0 \le 100-2\ell$

$2\ell \le 100$

$\ell \le 50$

$0 \le \ell \le 50$

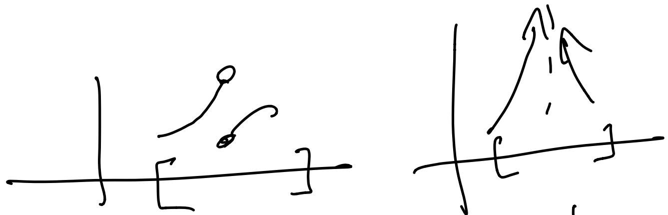
(or $0 < \ell < 50$)
philosophy $\Rightarrow$ we will ignore it.

The great thing about closed intervals as domains:

Theorem (Extreme Value)

If $f(x)$ is a continuous function whose domain is a closed interval then

- $f(x)$ always attains both its minimum & maximum values
- these always arise either at crit. pts or end pts.



can't happen.

strategy: to find extreme values, find all crit pts & plug crit pts & end pts into function
- biggest is max
- smallest is min.

Ex: $f(x) = \left(\sqrt[3]{x}\right)^2 + 1$ for x in $[-1, 8]$
is continuous everywhere and so also, here

crit pts: $f'(x) = 0$ or not defined $f(x) = x^{2/3} + 1$

$\frac{2}{3}\frac{1}{\sqrt[3]{x}} = \frac{2}{3}x^{-1/3} = f'(x)$
never $= 0$, but not defined at $x = 0$

crit pt @ $x = 0$ endpts $-1, 8$

$f(0) = \left(\sqrt[3]{0}\right)^2 + 1 = 1 \longleftarrow$ min
$f(-1) = \left(\sqrt[3]{-1}\right)^2 + 1 = 2$
$f(8) = \left(\sqrt[3]{8}\right)^2 + 1 = 2^2 + 1 = 5 \longleftarrow$ max.

Ex: $f(x) = \frac{x+1}{x-1}$ x in $[2, 3]$

$f'(x) = \frac{-2}{(x-1)^2}$ crit pts: none.

$f' < 0$ max at 2 min at 3