Def A Grothendieck topology on a category $\mathcal{C}$ is a collection of families of morphisms τ called "covers"

s.t. i) $\{a \xrightarrow{\sim} b\} \in \tau$

identity

ii) if $\{a_i \xrightarrow{f_i} b\} \in \tau$ and $c \in \mathcal{C}$

pullback

then $a_i \times_b c$ exist and $\{a_i \times_b c \to c\} \in \tau$

iii) if $\{a_i \xrightarrow{f_i} b\} \in \tau$ and $\forall i$, $\{c_{ij} \xrightarrow{g_{ij}} a_i\} \in \tau$

refinement

then $\{c_{ij} \xrightarrow{f_i \circ g_{ij}} b\} \in \tau$

ex: Top = cat of top spaces, set $\tau = \{\{a_i \xrightarrow{f_i} b\} \mid$ f_i gives a homeo between a_i & $f_i(a_i) \subset b$ open & $\{f_i(a_i)\}$ covers $b\}$

Zar/Sch = cat of schemes, $\tau = \{ \quad \}$ exactly as above but w/ f_i open immersion.

Zar/Rings^{op} same, restricted to affine schemes.

X top space, Open$(X) = \mathcal{C}$ $\quad \tau = \{\{U_i \to U\} \mid \text{cover}\}$

ex:
$$\begin{tikzcd} a \arrow[loop above] \arrow[dr] & \\ & c \arrow[loop right] \\ b \arrow[loop below] \arrow[ur] & \end{tikzcd}$$
$\tau: \{\{a \to c, b \to c\}, \text{id's} \ldots\}$

Rem: if X top spc, $\mathcal{C} = \mathrm{Open}(X)$

$$\begin{tikzcd} U_1 \arrow[r] & U \\ U_2 \arrow[ur] & \end{tikzcd} \qquad U_1 \times_U U_2 = U_1 \cap U_2$$

Def A site is a category w/ Groth. top. (definitions vary)

Def If $(\mathcal{C}, \tau)$ is a site, a presheaf (of sets) is a functor $\mathcal{C}^{op} \xrightarrow{\mathcal{F}} Sets$

we say $\mathcal{F}$ is a sheaf if for any $\{U_i \xrightarrow{f_i} U\} \in \tau$

(set $U_{ij} = U_i \times_U U_j$) we have

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

is an equalizer.

$\exists$ sheafification. i.e. given $\mathcal{F}$ a presheaf, $\exists$ $\hat{\mathcal{F}}$ a sheaf and a morphism $\mathcal{F} \longrightarrow \hat{\mathcal{F}}$ of presheaves

s.t. $\forall$ $\mathcal{F} \longrightarrow \mathcal{G}$, $\exists!$ $\hat{\mathcal{F}} \to \mathcal{G}$ s.t.

presheaf morphism   sheaf

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \hat{\mathcal{F}} \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathcal{G} \end{tikzcd}$$

<u>Categorical aside</u> (SGA 4.1 I.10 (Glossary) to Ch I)

<u>Def</u> $\mathcal{C}$ a cat, $f: a \to b$ is epic if

$$\mathrm{Hom}_{\mathcal{C}}(b,c) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(a,c) \text{ is injective for all } c$$

a family of morphisms $\{f_i : a_i \to b\}$ is epic if

$$\mathrm{Hom}_{\mathcal{C}}(b,c) \longrightarrow \prod \mathrm{Hom}_{\mathcal{C}}(a_i,c) \text{ is inj all } c.$$

$$\prod \mathrm{Hom}_{\mathcal{C}}(a_i,c) = \mathrm{Hom}_{\mathcal{C}}(\textstyle\coprod a_i, c) \text{ if } \textstyle\coprod a_i \text{ exists.}$$

a family $\{f_i : a_i \to b\}$ is a strict epi if it's an epi and if $(g_i) \in \prod \mathrm{Hom}(a_i,c)$ then (g_i) is in the image of $\mathrm{Hom}(b,c)$ if and only if

$\forall\, r \in \mathcal{C}$, all i,j, all φ_i, φ_j s.t.

$$\begin{tikzcd} r \arrow[r, "\varphi_i"] \arrow[dr, "\varphi_j"'] & a_i \arrow[d] \\ & a_j \arrow[r] & b \end{tikzcd} \quad \text{commute}$$

$\Rightarrow$ we have

$$\begin{tikzcd} r \arrow[r, "\varphi_i"] \arrow[d, "\varphi_j"'] & a_i \arrow[d] \\ a_j \arrow[r] & c \end{tikzcd} \quad \text{also commutes.}$$

Alternate formulation:

Let $\tilde{h}_a = \mathrm{Hom}_{\mathcal{C}}(a,-)$ $\left(h_a = \mathrm{Hom}_{\mathcal{C}}(-,a) \right)$

then $\{a_i \xrightarrow{f_i} b\}$ is epi $\iff$

$$\tilde{h}_b \to \prod \tilde{h}_{a_i}$$

is injective.

and $\{a_i \xrightarrow{f_i} b\}$ is strict epi

$\iff$

$$\tilde{h}_b \to \prod \tilde{h}_{a_i} \rightrightarrows \prod_{i,j} \tilde{h}_{a_i} \amalg_{\tilde{h}_b} \tilde{h}_{a_j}$$

is an equalizer.

$$\boxed{\tilde{h}_b \to \prod \tilde{h}_{a_i} \rightrightarrows \prod \tilde{h}_{a_i \times_b a_j}}$$

$$c \mapsto \mathrm{Hom}(a_i \times_b a_j, c)$$

$$\begin{tikzcd} a_i \arrow[r] \arrow[dr] & b \arrow[d, dashed] \\ & c \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r] \arrow[d] & a_i \arrow[d] \\ a_j \arrow[r] & b \end{tikzcd}$$

$$\begin{tikzcd} r \arrow[r] \arrow[d] & a_i \arrow[d] \\ a_j \arrow[r] & c \end{tikzcd}$$

~~need repair~~

$\Rightarrow$ $\{a_i \to b\}$ strict epi iff $\forall e$ we have

$$h_e(b) \longrightarrow \prod h_e(a_i) \rightrightarrows \prod h_e(a_i \times_b a_j)$$

is an equalizer

assuming these exist

Strict epi = satisfies the "sheaf condition" if $\{a_i \to b\}$ were a cover for the functor h_e.

Def $\{a_i \xrightarrow{f_i} b\}$ is universal epi (strict epi) if $\forall$ $c \to b$ $\{a_i \times_b c \longrightarrow c\}$ is an epi (strict epi)

Def $\mathcal{C}$ any cat, $can(\mathcal{C})$ "Canonical topology" is the cat s.t. covs are univ. strict epis.

= largest top s.t. rep. functors are sheaves.

$$\underset{b \mapsto h_b}{B \to} \underset{\text{Preshv}(B)}{\text{Fun}(B^{op}, \text{Sets})}$$

Thm (Giraud) $\mathcal{C}, \tau$

$$\text{Shv}(\mathcal{C},\tau) \hookrightarrow \text{Fun}(\text{Shv}(\mathcal{C},\tau)^{op}, \text{Sets})$$

gives an equiv of cats

$$\text{Shv}(\mathcal{C},\tau) \xrightarrow{\sim} \text{Shv}(\text{Shv}(\mathcal{C},\tau), \text{Can})$$

Def A topos is a cat $\mathcal{C}$ which is equiv to $Shv(B)$.
B some site.
Naturally consider such a topos w/ its can. topology.

Moral: really just need the notion of a sheaf.

Note: always have $B \longrightarrow Shv(B, Can)$

$$b \longmapsto h_b$$

Submoral: Sheaves are generalized objects.

$\mathcal{F}$ sheaf $\mathcal{G}$ a sheaf can (& should) consider

$$\mathcal{F}(\mathcal{G}) \text{ as } Hom(\mathcal{G}, \mathcal{F})$$

Yoneda (II) $\quad \mathcal{F} : \mathcal{C}^{op} \longrightarrow Sets.$

$X \in \mathcal{C} \qquad \mathcal{F}(X) \underset{nat}{=} Hom_{Fun}(h_X, \mathcal{F})$

formally define $\mathcal{F}(\mathcal{G}) = Hom_{Fun}(\mathcal{G}, \mathcal{F})$