

Def A Grothendieck topology on a category \mathcal{C} is a collection of families of morphisms τ called "covers"

s.t. i) $\{a \xrightarrow{\sim} b\} \in \tau$

identity pullback \nearrow ii) if $\{a_i \xrightarrow{f_i} b\} \in \tau$ and $c \in \mathcal{C}$
 then $a_i \times_b c$ exist and $\{a_i \times_b c \rightarrow c\} \in \tau$

rebeant \nearrow iii) if $\{a_i \xrightarrow{f_i} b\} \in \tau$ and $\forall i, \{c_{ij} \xrightarrow{g_{ij}} a_i\} \in \tau$
 then $\{c_{ij} \xrightarrow{f_i \circ g_{ij}} b\} \in \tau$

ex: Top = cat of top spaces, set $\tau = \{ \{a_i \xrightarrow{f_i} b\} \mid$
 f_i gives a homeo between a_i & $f_i(a_i)$ ch open & $\{f_i(a_i)\}$ cover b

Zar/Sch = cat of schemes, $\tau = \{ \}$ exactly as above but w/ f_i open immersions.

Zar/Rings same, restricted to affine schemes.

X top space, $\text{Open}(X) = \mathcal{C} \quad \tau = \{ \{U_i \rightarrow U\} \mid \text{cover} \}$

ex: $\begin{matrix} \circ & & \circ \\ \uparrow & & \uparrow \\ \circ & \xrightarrow{a} & \circ \\ \uparrow & & \uparrow \\ \circ & & \circ \end{matrix} \rightarrow \begin{matrix} \circ \\ \uparrow \\ \circ \end{matrix}$ $\tau: \{ \{a \rightarrow c, b \rightarrow c\}, \text{id's} \dots \}$

Remi: if X top spe, $\mathcal{C} = \text{Open}(X)$

$$\begin{array}{ccc} u_1 & \longrightarrow & u \\ & & \nearrow \\ u_2 & & \end{array} \quad u_1 \times_u u_2 = u_1 \cap u_2$$

Def A site is a category w/ Groth. top. (definitions vary)

Def If (C, τ) is a site, a presheaf (of sets)

is a functor $C^{op} \rightarrow \text{Sets}$

we say \mathcal{F} is a sheaf if for any $\{U_i \xrightarrow{f_i} U\}_{i \in I}$

(set $U_{ij} = U_i \times_U U_j$) we have

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

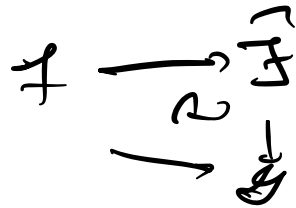
is an equalizer.

\mathcal{F} sheafification. i.e. given \mathcal{F} a presheaf, $\hat{\mathcal{F}}$ a sheaf

and a morphism $\mathcal{F} \rightarrow \hat{\mathcal{F}}$ of presheaves

cl. $\mathcal{F} \rightarrow \mathcal{G}$, $\mathcal{F} \rightarrow \hat{\mathcal{F}} \rightarrow \mathcal{G}$ s.t.

presheaf
morphism sheaf



Categorical aside (SGA 4.1 I.10 (Glossary))
to Ch I

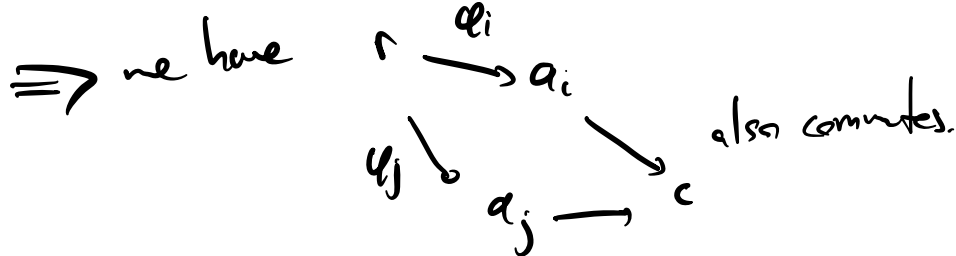
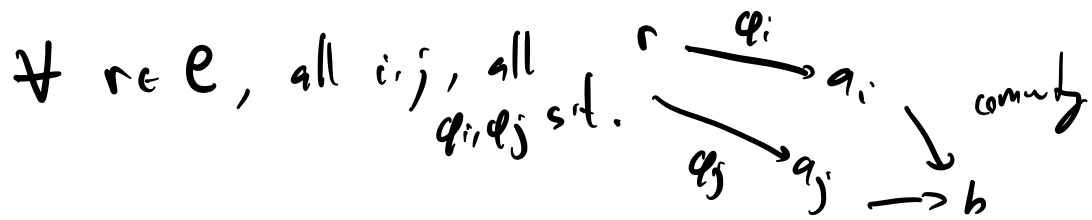
Def \mathcal{C} a cat, $f: a \rightarrow b$ is epic if
 $\text{Hom}_{\mathcal{C}}(b, c) \rightarrow \text{Hom}_{\mathcal{C}}(a, c)$ is injective for all c

a family of morphisms $\{f_i: a_i \rightarrow b\}$ is epic if

$\text{Hom}_{\mathcal{C}}(b, c) \rightarrow \prod \text{Hom}_{\mathcal{C}}(a_i, c)$ is inj for all c .

"
 $\text{Hom}_{\mathcal{C}}(\coprod a_i, c)$ if $\coprod a_i$ exists.

a family $\{f_i: a_i \rightarrow b\}$ is a strict epi if
 it's an epi and if $(g_i) \in \prod \text{Hom}(a_i, c)$
 then (g_i) is in the image of $\text{Hom}(b, c)$ if and only if



Alternate Lemma:

$$\text{let } \tilde{h}_a = \text{Hom}_C(a, -) \quad (h_a = \text{Hom}_C(-, a))$$

then $\{a_i \xrightarrow{f_i} b\}$ is epi \Leftrightarrow

$$\tilde{h}_b \rightarrow \prod \tilde{h}_{a_i}$$

is injective.

and $\{a_i \xrightarrow{f_i} b\}$ is strict epi

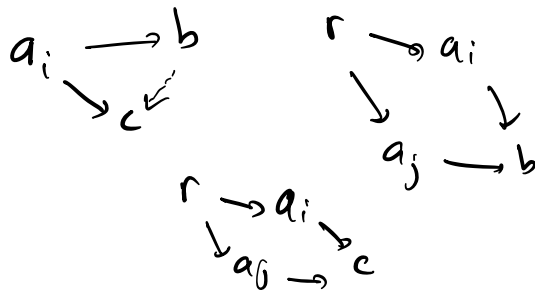
\Leftrightarrow

$$\tilde{h}_b \rightarrow \prod \tilde{h}_{a_i} \rightrightarrows \prod_{i,j} \tilde{h}_{a_i} \prod_{\tilde{h}_b} \tilde{h}_{a_j}$$

is an equalizer.

$$\tilde{h}_b \rightarrow \prod \tilde{h}_{a_i} \rightrightarrows \prod \tilde{h}_{a_i \times_b a_j}$$

$$c \mapsto \text{Hom}(a_i \times_b a_j, c)$$



~~need
reparion~~

$\Rightarrow \{a_i \rightarrow b\}$ strict epi iff $\forall c$ we have

$$h_c(b) \rightarrow \prod h_c(a_i) \xrightarrow{\cong} \prod h_c(a_i \times_b a_j)$$

is an equality ↑
assoc. flow exist

Strict epi = "sheaf condition" if $\{a_i \rightarrow b\}$ ^{cover} satisfies the \hat{c} for the functor h_c .

Def $\{a_i \xrightarrow{f_i} b\}$ is universal epi (strict epi) if $\forall c \rightarrow b$ $\{a_i \times_b c \rightarrow c\}$ is an epi (strict epi)

Def \mathcal{C} any cat, $\text{can}(\mathcal{C})$ "Canonical topology" is the cat s.t. covs are univ. strict epis.
= largest top s.t. rep. functors are sheaves.

Thm (Groth) \mathcal{C}, τ $\text{Preshev}(\mathcal{B})$
 $\mathcal{B} \rightarrow \text{Fun}(\mathcal{B}^{\text{op}}, \text{sets})$
 $b \mapsto h_b$

$\text{Shv}(\mathcal{C}, \tau) \hookrightarrow \text{Fun}(\text{Shv}(\mathcal{C}, \tau)^{\text{op}}, \text{sets})$
gives an equiv. of cats
 $\text{Shv}(\mathcal{C}, \tau) \xrightarrow{\sim} \text{Shv}(\text{Shv}(\mathcal{C}, \tau), \text{can})$

Def A topology is a cat \mathcal{C} which is equiv to $\text{Shv}(\mathcal{B})$.

\mathcal{B} some site.

Naturally consider such a topology w/ its (sn. topology).

Moral: really just need the notion of a sheaf.

Notes always have $\mathcal{B} \rightarrow \text{Shv}(\mathcal{B}, \text{Can})$
 $b \mapsto h_b$

Summary: Sheaves are generalized objects.

\mathcal{F} sheaf \mathcal{G} a sheaf can (if should) consider
 $\mathcal{F}(\mathcal{G})$ as $\text{Hom}(\mathcal{G}, \mathcal{F})$

Yoneda (II) $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$.

$X \in \mathcal{C}$ $\mathcal{F}(X) = \text{Hom}_{\text{Nat}}^{\text{Fun}}(h_X, \mathcal{F})$

family obj $\mathcal{F}(\mathcal{G}) = \text{Hom}_{\text{Fun}}(\mathcal{G}, \mathcal{F})$