Def A Zariski sheaf (1) is a functor

$\mathcal{F} : \underline{Sch}^{op} \longrightarrow \underline{Sets}$ s.t. $\mathcal{F}|_{Open(X)^{op}} \to \underline{Sets}$ is a sheaf for all X

X a scheme
Zariski top

Def A Zariski sheaf (2) is a functor

$\mathcal{F} : \underline{Comm.Rings} \longrightarrow \underline{Sets}$ s.t.

thought of as a functor on basic opens

$\mathcal{F}|_{\{R_f\}'s} \left(\overset{\vee}{=} \mathcal{F}|_{(\text{Basic opens on } Spec R)^{op}} \right)$

any R

is a $\mathcal{B}$-sheaf
($\mathcal{B}$ basic opens in $Spec R$)

Prop There is an isom of categories

$\underline{\text{Zariski Sheaf (1)}} \longrightarrow \underline{\text{Zariski Sheaf (2)}}$

$\mathcal{F} \longmapsto \mathcal{F}|_{\text{Basic opens}}$

Pf: (Idea) consider a auxiliary cat

$\underline{\text{Zariski Sheaf}}$ (1')

$\mathcal{F} : \underline{\text{Affine Schemes}} \longrightarrow \underline{Sets}$ s.t. $\mathcal{F}|_{Spec R}$ is a Zariski sheaf.

identification Shv $\longleftrightarrow$ B-Shv.

Zariski Sheaf 1 $\longleftrightarrow$ Zariski Sheaf 1'

since affines are a basis for opens for any scheme.

$$\mathrm{Shv}\, X \hookrightarrow \mathrm{Shv}(\text{Affines in } X)$$

B-sheaves

$$\hookrightarrow \mathrm{Shv}(\text{Basic Affines} \ldots)$$

Language: "Big" Zariski sheaves

Recall: $\mathrm{Sch} \hookrightarrow \mathrm{AffSch} \longrightarrow \mathrm{Spc} = \mathrm{Fun}(\mathrm{Rngs}, \mathrm{Sets})$

$\mathrm{Rngs}^{op} \xrightarrow{\sim} \mathrm{AffSpc}$

$$X \rightsquigarrow \underline{\mathrm{Rngs}} \longrightarrow \underline{\mathrm{Sets}}$$

$$R \longmapsto \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\, R, X)$$

$\left(\overset{\shortparallel}{\mathrm{Spec}\, S}\right)$

$$\left(\begin{array}{c} \shortparallel \\ \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\, R, \mathrm{Spec}\, S) \\ \shortparallel \\ \mathrm{Hom}_{\mathrm{Rngs}}(S, R) \end{array} \right)$$

$$\mathrm{Sch} \longrightarrow \mathrm{Spc}$$

$$\hookrightarrow \mathrm{ZariskiShv}\ (2)$$

$$\begin{tikzcd} \mathrm{Sch} \subset \text{Zariski}\,\mathrm{Shv}(2) \arrow[r, phantom, "\subset"] & \mathrm{Spc} \end{tikzcd}$$

$$\text{Zariski}\,\mathrm{Shv}(2) = \text{Zariski}\,\mathrm{Shv}(1)$$

$$\mathrm{Sch} \xrightarrow{?} \text{Zariski}\,\mathrm{Shv}(1)$$

$$X \longmapsto \mathrm{Hom}_{\mathrm{Sch}}(-, X)$$

sch.

Q: is a shaf $\mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set})$

$\cup$

if it is, $\mathrm{Sch} \hookrightarrow \mathrm{ZariShv}(1)$

is an imbedding (fully faithful)

by Yoneda.

Why is this a sheaf?

<u>Exercise</u>: If $(X, \mathcal{O}_X)$ & $(Y, \mathcal{O}_Y)$ (locally) ringed spaces.

then define a funct

$$\mathrm{Open}(X)^{op} \xrightarrow{M} \mathrm{Sets}$$

$$M(U) = \mathrm{Hom}_{(\text{loc.})\text{ ringed spaces}}\big((U, \mathcal{O}_X|_U), (Y, \mathcal{O}_Y)\big)$$

is a sheaf on X.

$\{U_i\}$ cov of U

$$M(U) \longrightarrow \prod M(U_i) \rightrightarrows \prod M(U_i \cap U_j)$$

maps $(U, \mathcal{O}_X|_U) \xrightarrow{f} (Y, \mathcal{O}_Y)$

determined by restrictions to cov $(U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{f|_{U_i}} (Y, \mathcal{O}_Y)$

& maps $f_i : (U_i, \mathcal{O}_X|_{U_i}) \longrightarrow (Y, \mathcal{O}_Y)$

s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ come from a global $f : (U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$

<u>Consequence</u>: if $(Y, \mathcal{O}_Y)$ is a scheme (and so a LRS)

then for any other scheme X

$$\begin{aligned} \mathrm{Open}(X)^{op} &\longrightarrow \mathrm{Sets} \\ U &\longmapsto \mathrm{Hom}_{Sch}(U, Y) \\ &\quad\; \underbrace{\mathrm{Hom}_{LRS}}((U, \mathcal{O}_X|_U), (Y, \mathcal{O}_Y)) \end{aligned}$$

is a sheaf (it's an "M")

$$\Rightarrow \quad \underline{Sch^{op}} \longrightarrow \underline{Sets}$$
$$X \longmapsto \mathrm{Hom}_{Sch}(X, Y)$$

is a Zariski sheaf (1)

$$y \longmapsto \mathrm{Hom}_{Sch}(-, y)$$

Z.Shv(1)

describes a functor $\underline{Sch} \to \mathrm{ZShf}(1) \overset{\text{fully faithful}}{\hookrightarrow} \mathrm{Fun}(\underline{Sch}^{op}, \underline{Sets})$

Yoneda

Yoneda $\Rightarrow$ $\underline{Sch} \to$ Z.Shf(1) fully faithful.

$$\begin{aligned} &\text{Z.Shf}(1) \\ &\parallel \\ &\text{Z.Shf}(2) \overset{\text{f.f.}}{\hookrightarrow} \mathrm{Fun}(\text{C.Rgs}, \text{Sets}) \\ &\qquad\qquad\qquad \parallel \\ &\qquad\qquad\qquad \underline{Spc}. \end{aligned}$$

also get $\underline{Sch} \to \underline{Spc}$ fully faithful.

$$\underline{Sch} \overset{\text{f.f.}}{\hookrightarrow} \underline{ZShv} \overset{\text{f.f.}}{\hookrightarrow} \underline{Spc}$$

$$\begin{tikzcd} \mathrm{ComRgs}^{op} = \mathrm{Aff\,Sch} \arrow[d] & = \mathrm{Aff\,Spc} \arrow[d, hook] \\ \underline{Sch} \arrow[r, hook, "\text{f.f.}"'] & \underline{ZShv} \arrow[r, hook, "\text{inclusion of subcat.}"'] & \underline{Spc} \end{tikzcd}$$

Gluing

Let X_1, X_2 top spaces $U_i \subset X_i$ open sets

$\varphi : U_1 \longrightarrow U_2$ homeomorphism.

$$X_1 \sqcup_\varphi X_2 \equiv \frac{X_1 \sqcup X_2}{\sim}$$

($\sqcup$ = disj union) $\qquad \sim = \{ (u_1, \varphi(u_1)) \mid u_1 \in U_1 \}$

new top space $X = X_1 \sqcup_\varphi X_2$ $\quad$ X has "local structure" inherited from X_1 & X_2.

If $\mathcal{F}_i$ shef on X_i & $\varphi^\#: \mathcal{F}_2|_{U_2} \longrightarrow \varphi_* \mathcal{F}|_{U_1}$

Can construct a shef $\mathcal{F}$ on $X = X_1 \sqcup_\varphi X_2$

new space X has open covs V_1, V_2 images of X_1 & X_2

and $U_1 \xrightarrow{\varphi} U_2 \Longleftrightarrow V_1 \cap V_2$ in X

$$\begin{tikzcd}
X \arrow[r, "\sim"] & X_1 \sqcup_\varphi X_2 \\
V_i \arrow[u] \arrow[r, "\sim"] & X_i \arrow[u, hook] \\
V_1 \cap V_2 \arrow[u, hook] \arrow[r, "\sim"] & U_1 \arrow[u]
\end{tikzcd} \qquad (\star)$$

From this (slightly abusive) perspective.

can consider $\mathcal{F}_i$ on V_i w/ $\mathcal{F}_2|_{V_1\cap V_2} \xrightarrow{\varphi^\#} \mathcal{F}_1|_{V_1\cap V_2}$

new sheaf $\mathcal{F}(U) = \left\{ (s_1,s_2) \in \mathcal{F}(U\cap V_1)\times\mathcal{F}(U\cap V_2) \;\middle|\; s_1|_{U\cap V_1\cap V_2} = s_2|_{U\cap V_1\cap V_2} \right\}$

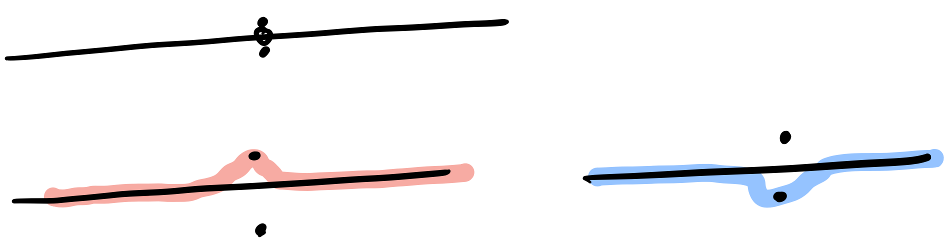
to check this is a sheaf,

Set $\mathcal{B} = \{ U \subset X \text{ open} \mid U\subset V_1 \text{ or } U \subset V_2\}$

basis.

$\varphi : U \xrightarrow{\sim} V$

$L_1 \sqcup_\varphi L_2$



In this way, given pairs $(X_1, \mathcal{O}_{X_1})$ $(X_2, \mathcal{O}_{X_2})$

and $U_i \subset X_i$ and $\varphi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \xrightarrow{\sim} (U_2, \mathcal{O}_{X_2}|_{U_2})$

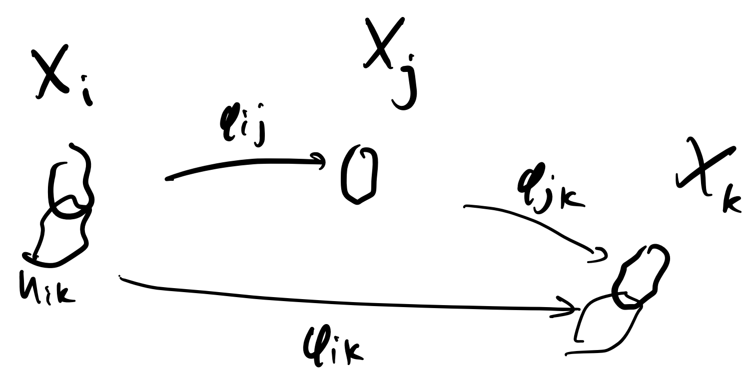
(loc) ringed spaces, can use above to glue to get

a new (loc.) ringed space.

If X_1, X_2 schemes, then so is $X_1 \sqcup_\varphi X_2$

More generally, if we have a collection $X_i \hookleftarrow U_{ij}$

and iso: $\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ with $U_{ij} \subset X_i$, $U_{ji} \subset X_j$



if $\varphi_{jk}\left(\varphi_{ij}|_{U_{ik} \cap \varphi_{ij}^{-1}(U_{jk})}\right) = \varphi_{ik}|_{U_{ik} \cap \varphi_{ij}^{-1}(U_{jk})}$ $(\ast\ast)$

1-cocycle condition

In this case, can glue X_i's to obtain X

s.t. analog of (☆) holds.

i.e. can identify X_i's w/ opens $V_i \subset X$

U_{ij}'s w/ $V_i \cap V_j$

<u>Gluing of sheaves</u> : $\mathcal{F}_i$ on $V_i \subset X$ open.

if have $\mathcal{F}_i|_{V_i \cap V_j} \xrightarrow{\psi_{ij}} \mathcal{F}_j|_{V_i \cap V_j}$

s.t. $\psi_{ik}|_{V_i \cap V_j \cap V_k} = \psi_{jk}|_{V_i \cap V_j \cap V_k} \circ \psi_{ij}|_{V_i \cap V_j \cap V_k}$

Given (loc.) ringed spaces $(X_i, \mathcal{O}_{X_i})$ isom's.

$$\varphi_{ij} : (\underset{X_i}{\underset{\cap}{U_{ij}}}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$$

s.t. (☆☆) cocycle holds. then $\exists$ (loc.) ringed

space $(X, \mathcal{O}_X)$ and a diagram like (☆)

of (L)RS

Gluing sheaves (executive summary)

there exists an equiv. of categories $\qquad \{V_i\}$ cover of X

$$\mathrm{Shv}_X \xrightarrow{\sim} \underset{\text{Desc}}{\text{Glue}}(\mathrm{Shv}, \{V_i\})$$

objects: $((\mathcal{F}_i), (\varphi_{ij}))$

s.t. $\mathcal{F}_i$ sheaf on V_i

$$\varphi_{ij} : \mathcal{F}_i|_{V_i \cap V_j} \to \mathcal{F}_j|_{V_i \cap V_j}$$

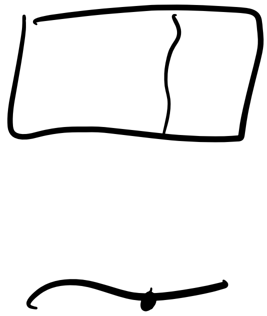
s.t. cocycle cond

$$\varphi_{ik}| = \varphi_{jk}| \circ \varphi_{ij}|$$

$\mathrm{Hom}\big(((\mathcal{F}_i), (\varphi_{ij})), ((\mathcal{G}_i), (\psi_{ij}))\big)$

" maps of sheaves $\rho_i : \mathcal{F}_i \to \mathcal{G}_i$

s.t.
$$\begin{tikzcd} \mathcal{F}_i|_{V_i \cap V_j} \arrow[r, "\rho_i|_{\dots}"] \arrow[d, "\varphi_{ij}"'] & \mathcal{G}_i|_{\dots} \arrow[d, "\psi_{ij}"] \\ \mathcal{F}_j| \arrow[r, "\rho_j|"'] & \mathcal{G}_j|_{\dots} \end{tikzcd}$$
commutes



$$\begin{tikzcd} X \arrow[d] & \cdots \\ S & S' \arrow[l] \end{tikzcd}$$