Last time: ended w/ def of (formally) smooth, unram., étale morphisms.

Today: plan is cohom / derived functors / idea of derived cats.

Later: derived ideas ...

Cohomology of sheaves

"Reminder" smooth manifolds

we learn a number of formulations of cohom

X sm. manifold   triangulation $H^n_{simp}(X, A)$ $\mathbb{Z}$ $\mathbb{R}$

$$H^n_{sing}(X, A)$$

deRham cohom $\Omega^0(X, \mathbb{R}) \xrightarrow{d} \Omega^1(X, \mathbb{R}) \xrightarrow{d} \cdots$

$C^\infty(X, \mathbb{R})$

$H^n_{dR}(X, \mathbb{R}) = $ Cohom of

(& these all agree)

Two features:

- Give rise to various LE sequences
- deRham computed w/ resolutions.

ses $\quad 0 \to A \to B \to C \to 0$

$$H^n(X,A) \to H^n(X,B) \to H^n(X,C) \to H^{n+1}(X,A) \to$$

Idea of today, if we generalize our notion of coefficients, then $H^i(X,-)$ are universally defined by this property

$A = $ Ab. gp

<u>Idea:</u> $n=0$ $\qquad H^0(X,A) = \Gamma(X,A)$

$$0 \to \Gamma(X,\underline{A}) \to \Gamma(X,\underline{B}) \to \Gamma(X,\underline{C}) \to R^1\Gamma \;?$$

$\Gamma(X,\underline{A}) = \mathrm{Maps}(X,A)$

$\underline{\mathbb{R}} \neq C^\infty(-,\mathbb{R})$

$\underline{A}$ "sheaf A" $= (U \mapsto \mathrm{Maps}^{\text{cont}}(U,A))$

<u>Computational method:</u> find a candidate for B w/ $A \hookrightarrow B$ & where ($?\ R^i\Gamma(X,B) = 0$, $i>0$)

★ $\quad \cdots \to \Gamma(X,\underline{B}) \to \Gamma(X,\underline{C}) \to \underline{R^1\Gamma(X,\underline{A})} \to 0$

Def $R^1\Gamma(X, \underline{A}) = \mathrm{coker}\left(\Gamma(X, \underline{B}) \to \Gamma(X, \underline{C})\right)$

$$\underline{C} = \underline{B}/\underline{A}$$

What ensures this $\exists$ B?

Contextual, but in smooth setting: soft

example: $\Omega^i(-, \underline{A})$ soft.

Main punchline of cohom:

we can compute $H^n(X, A)$ by choosing an exact seq of sheaves

$$\underline{A} \to S_0 \to S_1 \to S_2 \to \cdots$$

soft sheaves

i.e. then $H^n(X, \underline{A}) = H^n(\Gamma(S_0) \to \Gamma(S_1) \cdots)$

this is the def freed by consideration of ~~A~~

ex: $\underline{\mathbb{R}} \to \Omega^0(-, \mathbb{R}) \to \Omega^1(-, \mathbb{R}) \to \cdots$

exact seq. of sheaves.

Cohomology V.

Algebraic geometry

Fundamental problems in AG

1) given an invertible sheaf $\mathcal{L}$ on X, compute $\Gamma(X, \mathcal{L})$

2) find an algebraic analogue of H^n_{dR} or H^n_{sing} ...?

Focus on 1: main issue is working w/ correction to right exactness.

given q.coh sheaves $0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{M}' \to 0$

find correction terms to right exactness of global sections.

Standard homological framework:

Universal δ-functors:

given a functor (left exact)

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

additive functor between Abelian categories

(ex: $\mathcal{C}$ = sheaves of Ab. gps on X scheme
$\mathcal{D}$ = Ab. gps)

$F = \Gamma$

<u>Want</u>: a sequence of functors R^iF w/ $R^0F = F$ and such that

given ses $0 \to M'' \to M \to M' \to 0$ in $\mathcal{C}$

get a LES $0 \to R^0FM'' \to R^0FM \to \cdots$

$\to \cdots \quad R^{i-1}FM' \to R^iFM'' \to R^iFM$

and s.t. given a morphism of SES's

$$\begin{array}{ccccccccc} 0 & \to & M'' & \to & M & \to & M' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N'' & \to & N & \to & N' & \to & 0 \end{array}$$

get a morphism of LES's.

$$\begin{array}{ccccccc} \to & R^iFM'' & \to & R^iFM & \to & \cdots \\ & \downarrow & & \downarrow & & \\ \to & R^iFN'' & \to & R^iFN & \to & \cdots \end{array}$$

<u>Def</u> a δ-functor $\mathcal{C} \to \mathcal{D}$ is a seq. of functors $T^i : \mathcal{C} \to \mathcal{D}$ s.t. SES in $\mathcal{C} \rightsquigarrow$ LES in $\mathcal{D}$

i.e. get a functor $SES(\mathcal{C}) \xrightarrow[Seq(T)]{} LES(\mathcal{D})$

<u>Def</u> A δ-functor T is universal if $\forall$ other δ-functors $T' : \mathcal{C} \to \mathcal{D}$ and $f^0 : T^0 \to T'^0$ natural trans

$\exists!$ sequence of nat. trans. $f^i : T^i \to T'^i$ s.t.

$0 \to M' \to M \to M'' \to 0$

then

$$\begin{tikzcd}
T^i M \arrow[r] \arrow[d, "f^i M"'] & T^i M'' \arrow[r] \arrow[d, "f^i M''"] & T^{i+1} M' \arrow[d, "f^{i+1} M'"] \\
T'^i M \arrow[r] & T'^i M'' \arrow[r] & T'^{i+1} M'
\end{tikzcd}$$

commutes

univ. δ functors are unique.

$$\mathrm{Hom}_\delta(\overset{\mathrm{Univ}(T_0)}{\overset{\shortparallel}{T}}, T') = \mathrm{Hom}_{Fun}(T^0, T'^0)$$

<u>Def</u> A functor $F : \mathcal{C} \to \mathcal{D}$ is <u>effaceable</u> if $\forall c \in \mathcal{C}$ $\exists$ $c' \in \mathcal{C}$ and a mono $c \to c'$ s.t.

$F(c') = 0$

<u>Thm</u> (Groth.) If $T = \{T^i\}$ a δ functor, it is univ. if each T^i, $i > 0$ is effaceable.

Big observation: if $\mathcal{C}$ has enough injectives (i.e. all c admit $c \to c'$ c' injective object) then univ. δ-functors exist.

$\mathcal{L}$ inv. sheaf $\quad \Gamma(\mathcal{L})$

$$0 \to \mathcal{K} \to \mathcal{M} \twoheadrightarrow \mathcal{L} \to 0$$

$$0 \to \Gamma(\mathcal{K}) \to \Gamma(\mathcal{M}) \to \underset{\substack{T^0(K \\ \shortparallel}}{\Gamma(\mathcal{L})} \to \underset{\substack{T^1(K) \\ \nearrow}}{R^1\Gamma(\mathcal{K})}$$

notation: if T is a univ. delta functor w/ $T^0 = F$ we call T^i's the ith satellite or i^{th} right derived functor of F, write it $R^iF = T^i$

Small annoyance: we are interested in Γ of coherent (& occasionally q-coh.) sheaves. Nice Ab cat.

But - not enough injectives.

Fix: $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X) \to \mathcal{O}_X\text{-mod}$

enough injectives

can define right derived functors

here.

Def (Sheaf cohomology) $\mathcal{F}$ a shf of $\mathcal{O}_X$-mods

$$H^n(X,\mathcal{F}) \equiv R^n\Gamma(\mathcal{F})$$

Concretely, can compute as follows: exact

choose $\mathcal{F} \hookrightarrow \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$

$\mathcal{I}_i$ injective shf

$$H^n(X,\mathcal{F}) = H^n(\Gamma(\mathcal{I}_\bullet))$$

What does topology have to do with this?

Cohom is a correction factor for surjectivity of sheaves $\rightsquigarrow$ surj of global sections.

Topology $\rightsquigarrow$ natural f surj of sheaves.

X = some variety over $\mathbb{C}$

$$\mathcal{O}_X^* \xrightarrow{\cdot 2} \mathcal{O}_X^*$$

sheaf of Ab. gps

this is not Zariski surjective. $f \in \mathcal{O}_X^*(U)$ but is surj. in standard analytic top.

$X = \operatorname{Spec} k[t]$

$U \subset \mathbb{A}^1 \setminus \{0\}$

$f = t$

$\sqrt{t} \in k(t)$ not.

<u>Def</u> X a scheme, $X_{\text{ét}}$ is the Groth. top on cat w/ objects

$U \xrightarrow{f} X$ f étale and covers

$\{U_i \xrightarrow{f_i} X\}$ f_i étale & jointly surj.

$\bigcup$ scheme-theoretic images $= X$

<u>Punchline</u>: much closer to analytic — étale surj.

v. close to an. surj.

(e.g. sm. q. proj vars)

<u>Thm 2</u>: δ functors useful, but philosophically inadequate.

<u>Better perspective</u>:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & H^i(X) \\ & M(X) \arrow[ur] & \end{tikzcd}$$

derived motive → $M(X)$

abelian thy.

$$\begin{tikzcd} \text{Sheves}/X \arrow[rr, squiggly] \arrow[dr] & & H^\bullet(\ ) \\ & \text{complex/derived cat. of sheaves} \arrow[ur] & \end{tikzcd}$$

<u>Verdier</u>

$$\cancel{\mathcal{F} \to} \underbrace{\mathcal{I}_0 \to \mathcal{I}_1 \to \cdots}_{\Gamma}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathcal{I}_0 \arrow[r] & \mathcal{I}_1 \arrow[r] & \mathcal{I}_2 \end{tikzcd}$$

$$R\Gamma(\mathcal{F}) = \left(\Gamma(\mathcal{I}_0) \to \Gamma(\mathcal{I}_1) \to \cdots \right)$$

$$\begin{tikzcd} \text{Shves} \arrow[r, "F"] \arrow[d, hook] \arrow[ddr, bend right] & Ab \\ \text{Com}(\text{Shvs})/\sim \arrow[r, "RF"] & \text{Com}(Ab)/\sim \arrow[d, "\text{coh.}"] \\ & R^i F \; Ab \end{tikzcd}$$