Plan: today & next time: invertible sheaves, divisors, maps to proj. space.

a few words about blowing up.

- Formal schemes, gluing, adic spaces
- Differentials / smooth morphisms, étale morphisms etc.

$\rightsquigarrow$ cotangent complex $\rightsquigarrow$ derived schemes?

- cohomology étale cohom. $\rightsquigarrow$ étale topology

Last time: for any A, an A-scheme X

we saw $\mathrm{Hom}_{A\text{-sch}}(X, \mathbb{P}^n_A) = \widetilde{\mathbb{P}}^n_A(X)$

$$= \{ \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L} \mid \mathcal{L} \text{ inv. sheaf} \} / \sim$$

$$\begin{array}{ccc} \mathcal{O}_X^{n+1} & \twoheadrightarrow & \mathcal{L} \\ \| & & \downarrow \wr \\ \mathcal{O}_X^{n+1} & \twoheadrightarrow & \mathcal{L}' \end{array}$$

can think of a morphism $\varphi : X \longrightarrow \mathbb{P}^n_A$ as an inv sheaf $\mathcal{L}$ & tuple of global sections $s_0, \ldots, s_n \in \Gamma(\mathcal{L})$

$$x \longmapsto [s_0(x) : \ldots : s_n(x)]$$

$\varphi : X \longrightarrow \mathbb{P}^n_A$ defined by $h_\varphi : h_X \to h_{\mathbb{P}^n_A}$

functors either in Sch/A or $\underline{(A\text{-alg})^{op}}$ $\quad h_{\mathbb{P}^n_A} = \mathbb{P}^n_A$

$$h_X(B) \longrightarrow h_{\mathbb{P}^n_A}(B) = \mathbb{P}^n_A(B)$$

$$X(B) = h_X(B)$$

$$\mathbb{P}^n_A(B) = \{\text{proj. rk } 1 \ P \text{ w/ } B^n \twoheadrightarrow P\} / \sim$$

(if X zeros of $f_1, \ldots, f_N$ in vars $T_1, \ldots, T_M$ i.e. $X = \operatorname{Spec} A[T_1, \ldots, T_M]/(f_1, \ldots, f_N)$

then $X(B) = \{\vec{b} = (b_1, \ldots, b_M) \in B^M \mid f_i(\vec{b}) = 0 \text{ all } i\}$

$A[T_1, \ldots]/(\vec{f}) \to B$)

given $\varphi : X \longrightarrow \mathbb{P}^n_A$ from $\mathcal{O}_X^{n+1} \overset{\pi}{\twoheadrightarrow} \mathcal{L}$ and given

$b \in X(B)$ $\big(b : \operatorname{Spec} B \longrightarrow X \big)$ $\quad b^* \mathcal{O}_X^{n+1} \xrightarrow{b^*\pi} b^* \mathcal{L} = \tilde{P}$

$b^* \mathcal{O}_X^{n+1} \cong \mathcal{O}_{\operatorname{Spec} B}^{n+1}$

corresp. to $B^{n+1} \twoheadrightarrow P$

Remark: Can globalize this:

for $\mathcal{E}/S$ rk $n+1$ loc. free shf of $\mathcal{O}_S$-modules

$$\begin{array}{ccc} \text{Consider } \underline{\mathrm{Proj}}_{\mathcal{O}_S} \mathrm{Sym}^\bullet_{\mathcal{O}_S} \mathcal{E} & \longleftarrow & \mathrm{Proj}\, \underbrace{\mathrm{Sym}^\bullet_A (A^{n+1})}_{A[x_0, \dots, x_n]} = \mathbb{P}^n_A \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}\, A \end{array}$$

if $\mathrm{Spec}\, A \subset S$ affine open s.t. $\mathcal{E}|_{\mathrm{Spec}\, A}$ is free

Set $\widetilde{\mathbb{P}}_S(\mathcal{E})(X) = \{ \mathcal{E} \twoheadrightarrow \mathcal{L} \mid \mathcal{L} \text{ invertible} \} / \sim$

$$\sim \text{ via } \begin{array}{ccc} \mathcal{E} & \twoheadrightarrow & \mathcal{L} \\ \| & & \downarrow \wr \\ \mathcal{E} & \twoheadrightarrow & \mathcal{L}' \end{array} \qquad \mathcal{O}_U^{n+1} \twoheadrightarrow \mathcal{L}$$

then (claim) $\widetilde{\mathbb{P}}_S(\mathcal{E})$ is a shef and

if $\mathcal{B}$ = cat of open affines in S s.t. $\mathcal{E}|_U$ is free

then $\widetilde{\mathbb{P}_S(\mathcal{E})}|_{\mathcal{B}} = h_{\underline{\mathrm{Proj}}_{\mathcal{O}_S} \mathrm{Sym}^\bullet_{\mathcal{O}_S} \mathcal{E}}|_{\mathcal{B}}$

$$\Longrightarrow \widetilde{\mathbb{P}}_S(\mathcal{E}) = h_{\underline{\mathrm{Proj}}_{\mathcal{O}_S} \mathrm{Sym}^\bullet_{\mathcal{O}_S} \mathcal{E}}$$

Def $\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_{\mathcal{O}_S} \mathrm{Sym}^{\bullet}_{\mathcal{O}_S} \mathcal{E}$

if $\mathcal{E}$ is loc. free not free $\mathbb{P}(\mathcal{E}) \neq \mathbb{P}^n$

$\mathcal{E} \to \mathbb{P}(\mathcal{E})$

if $\mathcal{E}$ free then $\mathcal{E} \simeq \mathcal{O}_X^n$ $\quad E \simeq A^{n+1}$

$\mathcal{E} = \tilde{E}$

$\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}^n_A$

Q: How to construct maps $X \longrightarrow \mathbb{P}^n_A$?

Ans: need line bundle $\mathcal{L}$ & global sections.

given $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$

$\mathcal{O}_X^{n+1} \longrightarrow \mathcal{L}$ consider for any s_i its vanishing locus.

$\mathcal{L} \not\simeq \mathcal{O}_X$ but locally $\mathcal{L} \simeq \mathcal{O}_X$.

choose $s \in \Gamma(\mathcal{L})$, then on any U w/ $\mathcal{L}|_U \simeq \mathcal{O}_U$

choosing an iso $\varphi : \mathcal{L}|_U \longrightarrow \mathcal{O}_U$ can consider ideal cut out by $\varphi(s_i)$ and this doesn't depend on φ!

$\mathcal{I}_{s_i}$ = gen on $\mathcal{U}$ by $\psi(s_i) \in \mathcal{O}_X(\mathcal{U}) = \mathcal{O}_{\mathcal{U}}(\mathcal{U})$

$$\begin{tikzcd} \mathcal{L}|_{\mathcal{U}} \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & \mathcal{O}_{\mathcal{U}} \\ & \mathcal{O}_{\mathcal{U}} \arrow[u, "\text{mult by } r \in \mathcal{O}_{\mathcal{U}}(\mathcal{U})^*"'] \end{tikzcd}$$

i.e. $\psi(s_i) = r \cdot \psi'(s_i)$

$s \in \Gamma(\mathcal{L})$ vanishes at $P \in X$ means that via ψ, $\psi(s)_p \in m_{X,p}$

or $s_p \in m_p \mathcal{L}_p$

$$\mathcal{L}|_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}$$

$$\mathcal{L}_p \xrightarrow{\sim} \mathcal{O}_{X,p}$$

$$m_p \mathcal{L}_p \leftarrow \text{--} \; m_p \mathcal{O}_{X_p}$$

<u>Notice:</u> $Z(s_i)$ (subscheme corresp to $\mathcal{I}_{s_i}$) is closed since its locally closed.

and if $p \notin Z(s_i)$ then $\mathcal{O}_{X,p} \to \mathcal{L}_p \simeq \mathcal{O}_{X_p}$, $1 \mapsto (s_i)$

is surjective

Def if all s_i's vanish at p we say p is a basepoint of $\{s_0, \dots, s_n\}$

if $U = X \setminus \{\text{basepts}\}$ U is open and
$$\{\text{basepts}\} = \bigcap Z(s_i)$$

we have a surjection $\mathcal{O}_U^{n+1} \twoheadrightarrow \mathcal{L}|_U$

get a morphism $U \longrightarrow \mathbb{P}^n_A$

Later, we'll see given $s_0, \dots, s_n \in \Gamma(\mathcal{L})$

then $\exists\, \tilde{X} \longrightarrow X$ (blowup) s.t.

have a morphism (proper birat'l)

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\pi"'] & \mathbb{P}^n_A \\ X \supset_j U \arrow[ur, "\varphi"'] & \end{tikzcd}$$

and w/ $\pi^{-1}(U) \simeq U$

$$\boxed{\text{Can globalize } \mathcal{E} \to \mathcal{L} \text{ !}}$$

Divisors

Recall: if R a ring, $r \in R$ regular if $r \neq 0$ &

$ar = 0 \Rightarrow a = 0$.

Lem $r \in R$ regular $\Longleftrightarrow$ $\frac{r}{1} \in R_{\mathfrak{p}}$ regular all $\mathfrak{p} \in \operatorname{Spec} R$

Pf: $(\Rightarrow)$ if $\frac{r}{1}\frac{a}{s} = 0$ w/ $s \notin \mathfrak{p}$ then $rat = 0$ some $t \notin \mathfrak{p}$

$\Rightarrow$ (r regular) $at = 0 \Rightarrow \frac{a}{1} = 0$ in $R_{\mathfrak{p}}$ $\Rightarrow \frac{a}{s} = 0$ ✓

$(\Leftarrow)$ if $\frac{r}{1}$ regular all $\mathfrak{p}$, suppose $ar = 0$.

then $\frac{a}{1} \cdot \frac{r}{1} = 0 \Rightarrow \frac{a}{1} = 0$ in $R_{\mathfrak{p}}$ $\Rightarrow \exists\, t \notin \mathfrak{p}$, $ta = 0$

$\Rightarrow \operatorname{ann}_R(a) \not\subset \mathfrak{p}$ any $\mathfrak{p}$. $\Rightarrow \operatorname{ann}_R(a) = R$

$1 \cdot a = a = 0$. ✓

Def The total ring of fractions of R, $Q(R) = R[S^{-1}]$

$S = \{\text{regular elements}\}$. $R \hookrightarrow Q(R)$.

Def for a scheme X, set $\mathcal{K}_X$ to be the sheaf given by

$\mathcal{K}_X(U) = \overbrace{Q(\mathcal{O}_X(U))}$ ~~(this is a sheaf)~~

sheafification of the (separated) presheaf above.

K_X is nice because in nice circumstances, invertible sheaves are subsheaves of K_X.

Affine case, we're saying often a P a rank 1 proj over R, can embed $P \hookrightarrow Q(R)$

<u>Prop</u> If X is integral (and so K_X = constant) then any $\mathcal{L}$ invertible is $\simeq$ to a $\mathcal{O}_X$-mod subsheaf of K_X.

<u>Pf</u>: Consider $\mathcal{L} \otimes_{\mathcal{O}_X} K_X$

locally the map $\mathcal{L} \xrightarrow{\iota} \mathcal{L} \otimes_{\mathcal{O}_X} K_X$, $s \longmapsto s \otimes 1$ looks like

$$\mathcal{L}|_U \xrightarrow{\iota|_U} \mathcal{L}|_U \otimes_{\mathcal{O}_U} K_U \qquad \mathcal{L}_U \simeq \mathcal{O}_U$$

$$\mathcal{O}_U \longrightarrow \mathcal{O}_U \otimes_{\mathcal{O}_U} K_U$$

$$\mathcal{O}_U \longrightarrow K_U \text{ injective}$$

$\Rightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes K_X$ loc. inj $\Rightarrow$ inj. (sheaf)

Claim: $\mathcal{L} \otimes_{\mathcal{O}_X} K_X \simeq K_X$

both sides are global sections (objects) in the stack <u>$\mathcal{O}_X$-mod</u>

choose $\{U_i\}$ cov s.t. $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$

$$\underset{\sim\sim}{\mathcal{O}_X\text{-mod}}(X) \xrightarrow{\sim} \mathrm{Desc}(\{U_i\}, \underset{\sim\sim}{\mathcal{O}_X\text{-mod}})$$

$$\overset{\shortparallel}{\underline{\mathcal{O}_X\text{-mod}}}$$

$$\mathcal{L} \longmapsto ((\mathcal{L}|_{U_i})_i, (1_{ij}))$$

$$(\mathcal{O}_{U_i}$$

$$\mathcal{L}|_{U_i}/U_i$$

$U_i \cap U_j$ iso $1_{ij} : \mathcal{L}|_{U_i}|_{U_{ij}} \to \mathcal{L}_{U_j}|_{U_{ij}}$

$$\overset{\shortparallel}{\mathcal{L}|_{U_{ij}}} \qquad \overset{\shortparallel}{\mathcal{L}|_{U_{ij}}}$$

$$\mathcal{L}|_{U_i} \xrightarrow{\varphi_i} \mathcal{O}_{U_i}$$

$$\begin{array}{ccc}
\mathcal{L}|_{U_i}|_{U_{ij}} & \xrightarrow{1} & \mathcal{L}|_{U_j}|_{U_{ij}} \\
\varphi_i|_{ij} \downarrow & & \downarrow \varphi_j|_{ij} \\
\mathcal{O}_{U_{ij}} = \mathcal{O}_{U_i}|_{U_{ij}} & \xrightarrow[\underbrace{\varphi_j|_{ij} \circ \varphi_i|_{ij}^{-1}}]{} & \mathcal{O}_{U_j}|_{ij} = \mathcal{O}_{U_{ij}}
\end{array}$$

$$\varphi_{ij} \in \mathrm{Iso}_{\mathcal{O}_{U_{ij}}}(\mathcal{O}_{U_{ij}}, \mathcal{O}_{U_{ij}}) \simeq \mathcal{O}_X(U_{ij})^*$$

$$(\{\mathcal{L}|_{U_i}\}, \{1\}) \simeq (\{\mathcal{O}_{U_i}\}, \varphi_{ij})$$
in $\mathrm{Desc}(\{U_i\})$

similarly, we find

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X \rightsquigarrow (\{\mathcal{L} \otimes \mathcal{K}_X|_{U_i}\}, \{1\})$$
$$\wr\!\!\wr$$
$$(\{\mathcal{K}_{U_i}\}, \varphi_{ij})$$
$$\varphi_{ij} \in \mathcal{O}_X^*(U_{ij}) \subset \mathcal{K}_X^*(U_{ij})$$

$$\mathcal{K}_{U_{ij}} \xrightarrow{\cdot \varphi_{ij}} \mathcal{K}_{U_{ij}}$$

<u>Claim:</u> $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X \simeq \mathcal{K}_X$ via

$$(\{\mathcal{K}_{U_i}\}, \varphi_{ij}) \simeq (\{\mathcal{K}_{U_i}\}, 1)$$

Desc

<u>Notice:</u> $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ in $\mathcal{K}_X(U_{ijk})^*$ all i,j,k (think of this as an eqn in $K_X = \Gamma(\mathcal{K}_X) = \mathcal{K}_X(U)$)

have $\varphi_{ii} = \varphi_{ii}\varphi_{ii}$

$\Rightarrow \varphi_{ii} = 1$

$$1 = \varphi_{ii} = \varphi_{ji}\varphi_{ij} \Rightarrow \varphi_{ji} = \varphi_{ij}^{-1}$$

$$\varphi_{ij} = \varphi_{1j}\,\varphi_{i1} = \varphi_{1j}\,\varphi_{1i}^{-1}$$

Define a map

$$\begin{tikzcd} (\{K_{U_i}\}, \varphi_{ij}) & K_{U_i} \\ (\{K_{U_i}\}, 1) \arrow[u] & K_{U_i} \arrow[u, "\cdot \varphi_{1i}"'] \end{tikzcd}$$

$$\begin{tikzcd} K_{U_i}|_{ij} \arrow[r, "\varphi_{ij} = \varphi_{1j}\varphi_{1i}^{-1}"] & K_{U_j}|_{ij} \\ K_{U_i}|_{ij} \arrow[u, "\varphi_{1i}"] \arrow[r, "1"'] & K_{U_j}|_{ij} \arrow[u, "\varphi_{1j}"'] \end{tikzcd} \Rightarrow K_X \simeq \mathcal{L} \otimes_{\mathcal{O}_X} K_X.$$

$\square$

X integral, can actually see $K_X = i_* \mathcal{O}_{\operatorname{Spec} k(X)}$

$k(X)$ = ff. of X

$\operatorname{Spec} k(X) \xrightarrow{i} X$ inclusion of generic pt.

$\operatorname{Spec} k(X) \longrightarrow \operatorname{Spec} A$

$K_X = k(X) \longleftarrow A$

(Nakai: still true for X proj / field)
(check adjectives)

$$f_i \longleftarrow s_i \longleftarrow 1$$

$\mathcal{L} \hookrightarrow \mathcal{K}_X$ locally $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}$ gen by a single element

choose $\{U_i, f_i\}$ $f_i \in \mathcal{K}_X(U)$ gen (i.m.f) $\mathcal{L}$

$f_i|_{U_{ij}}$, $f_j|_{U_{ij}}$ gen same subsheaf of $\mathcal{K}_{U_{ij}}$

$\Rightarrow f_i = u_{ij} f_j$ $u_{ij} \in \mathcal{O}_X(U_{ij})^*$

it makes sense to consider $\mathrm{div}(f_i)$ (zeros & poles)

i.e. $(U_i, f_i\} \rightsquigarrow$ locally principal Weil divisor.

<u>Def</u> A Cartier divisor is a collection (U_i, f_i)
U_i cover $f_i \in \mathcal{K}_X(U_i)^*$ s.t.
$f_i = u_{ij} f_j$ on U_{ij} $u_{ij} \in \mathcal{O}_X(U_{ij})^*$
/ $\sim$

via refining open set
&c.

$(U_i, f_i) \sim (U_i, g_i)$ if

$f_i = v_i g_i$ v_i unit.