Sheaves of $\mathcal{O}_X$-modules

Def a sheaf of $\mathcal{O}_X$-mods is a sheaf of Ab gps $\mathcal{M}$ w/ $\mathcal{M}(U)$ having structure of an $\mathcal{O}_X(U)$-module s.t. $m \in \mathcal{M}(U)$, $r \in \mathcal{O}_X(U)$ $V \subset U$ then

$$(rm)|_V = r|_V \cdot m|_V.$$

Remark: We discussed that the Cat. of Abelian sheaves has kernel, cokernels, quotients. These carry over to $\mathcal{O}_X$-mod

$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ = morphisms of sheaves $\mathcal{M} \to \mathcal{N}$ s.t. $\forall U$ $\mathcal{M}(U) \to \mathcal{N}(U)$ is an $\mathcal{O}_X(U)$-module hom.

is an $\mathcal{O}_X(X)$ mod

Also: Internal homs: i.e. given $\mathcal{M}, \mathcal{N}$ $\mathcal{O}_X$-modules, can form

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ an $\mathcal{O}_X$-mod

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U)$$

$$\overset{?}{=} \cancel{\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{M}(U), \mathcal{N}(U))} \; ?$$

$$\mathcal{M} \to \mathcal{M} \oplus \mathcal{M}$$

__Def__ $\mathcal{H}om^{pre}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})(U) = \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{M}(U),\mathcal{N}(U))$

__Claim:__ ~~$\mathcal{H}om$ = sheafification of $\mathcal{H}om^{pre}$.~~

~~$\mathcal{H}om \longrightarrow \mathcal{H}om^{pre}$~~

bad idea

$V \subset U$

$$\begin{tikzcd} \mathcal{M}(U) \arrow[r] \arrow[d] & \mathcal{N}(U) \arrow[d] \\ \mathcal{M}(V) \arrow[r, dashed] & \mathcal{N}(V) \end{tikzcd}$$

Suppose $(X,\mathcal{O}_X)$ scheme, $\mathcal{A}$ a sheaf of $\mathcal{O}_X$-algebras.

i.e. $\mathcal{A}(U)$ an $\mathcal{O}_X(U)$-mod also w/ algebra structure

s.t. $\mathcal{A}$ a sheaf of algebras.

Can talk about: sheaves of $\mathcal{A}$-modules

kernels, cokernels, internal homs

$\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N}) = \{$ morphs of functors $\mathcal{M}\to\mathcal{N}$ s.t. $\mathcal{M}(U)\to\mathcal{N}(U)$ is an $\mathcal{A}(U)$-mod map $\}$

$\mathcal{M},\mathcal{N}$ $\mathcal{A}$-mods

$\mathcal{H}om_{\mathcal{A}}(\mathcal{M},\mathcal{N})(U) = \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{M}|_U,\mathcal{N}|_U)$

If $\mathcal{M}$ a right $\mathcal{A}$-mod, $\mathcal{N}$ a left $\mathcal{A}$-mod,

can form $\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}$ a sheaf of Ab. gps

(also an A-mod if A is comm.)

$$\mathcal{M} \overset{pre}{\otimes}_A \mathcal{N}(U) = \mathcal{M}(U) \otimes_{A(U)} \mathcal{N}(U)$$

$$\mathcal{M} \otimes_A \mathcal{N} = \widetilde{\mathcal{M} \overset{pre}{\otimes}_A \mathcal{N}} \quad \text{sheafification.}$$

Standard Adjunction $\quad A$ comm.

$$\mathcal{H}om_A(\mathcal{M} \otimes_A \mathcal{N}, \mathcal{P}) = \mathcal{H}om_A(\mathcal{M}, \mathcal{H}om_A(\mathcal{N}, \mathcal{P}))$$

$$(\mathcal{M} \otimes_A \mathcal{N})_x = (\mathcal{M} \overset{pre}{\otimes}_A \mathcal{N})_x \overset{?}{=} \mathcal{M}_x \otimes_{A_x} \mathcal{N}_x$$

$$= \varinjlim_{U \ni x} \mathcal{M}(U) \otimes_{A(U)} \mathcal{N}(U)$$

$$\begin{tikzcd}
\varinjlim \mathcal{M}(U) \otimes_{A(U)} \mathcal{N}(U) \arrow[r] \arrow[dr, "?", "\sim"'] & \mathcal{M}_x \otimes_{A(U)} \mathcal{N}_x \arrow[d] \\
& \mathcal{M}_x \otimes_{A_x} \mathcal{N}_x
\end{tikzcd}$$

$$m \otimes n \mapsfrom m \otimes n$$

$$0 \mapsfrom ma \otimes n - m \otimes an$$

If $f: X \longrightarrow Y$ morphism of ringed spaces

then get $f^*: \underline{\mathcal{O}_Y\text{-mod}} \longrightarrow \underline{\mathcal{O}_X\text{-mod}}$

$f_*: \underline{\mathcal{O}_X\text{-mod}} \longrightarrow \underline{\mathcal{O}_Y\text{-mod}}$

$\mathcal{M}$ an $\mathcal{O}_X$-mod, $f_*\mathcal{M}$ is naturally an $f_*\mathcal{O}_X$-mod

$f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ via $f^\#$ get an $\mathcal{O}_Y$-mod

backwards if $\mathcal{N}$ an $\mathcal{O}_Y$-mod then $f^{-1}\mathcal{N}$ is an $f^{-1}\mathcal{O}_Y$ mod.

$f^\# \longleftrightarrow f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$

define $f^*\mathcal{N} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N}$

<u>ex:</u>

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \widetilde{M \otimes_A B} & & \widetilde{M} \end{array} \qquad \begin{array}{c} A \longrightarrow B \\ M \end{array}$$

Proj   Affine   Projective.

Axiomatize notion of a homogeneous coord ring

$k[x_0, \dots, x_n]$ hom. coord ring of $\mathbb{P}^n$

S graded ring.

consider $U \subset \mathbb{P}^n$ defined by $x_1 \neq 0$ in U,

x_2/x_1 is a well defined fcn.

$f(P) \Leftrightarrow f(\lambda P)$

if f is hom. deg d

$f(\lambda P) = \lambda^d f(P)$

can make sense of values for $f(P)$ if f is hom. deg 0.

although may not have global fcns of this form

have lots of locally defined fcns of this sort.

Sheaf = repository of gen fcns.

<u>Def</u> (preview) an invertible sheaf (line bundle) is a sheaf of $\mathcal{O}_X$ modules, loc. $\cong$ to $\mathcal{O}_X$.

$k[x_0, \ldots, x_n] \longrightarrow \mathbb{P}^n$

hom. ideals, except $(x_0, \ldots, x_n)$ $\longleftrightarrow$ closed sets

(recall S a graded ring means $S \cong \bigoplus_{n \geq 0} S_n$ or $\bigoplus_{n \in \mathbb{Z}} S_n$

$I \triangleleft S$ hom $\equiv$ $I \cong \bigoplus I_n$

$I_n = I \cap S_n$

$\iff$ I gen. by hom elements)

$\mathrm{Proj}\, S = \{$ hom. prime ideals $\mathfrak{p} \triangleleft S$ which don't contain $S_{>0}$ $\}$

$S_{>0} = \bigoplus_{i=1}^{\infty} S_i$

Structure sheaf defined locally on basic open sets.

for $f \in S_d$ define D_f^+ $\quad D_+(f)$

$D_f^+ = \{$ hom primes don't contain f $\}$

$S_f = S[f^{-1}]$ $\quad$ $= \{$ hom primes of S_f $\}$

Set $\mathcal{O}_{\mathrm{Proj}\, S}(D_f^+) = S_{(f)} \equiv (S_f)_0$

Prop $\operatorname{Proj} S \supset D_f^+ \simeq \operatorname{Spec} S_{(f)}$

i.e. Proj is these Specs stuck together.

Ex: R comm. ring. $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$

has a cover by basic opens $\mathbb{A}^{n,i}_R = D^+_{x_i}$

$= \operatorname{Spec} R[x_0/x_i, \dots, x_n/x_i]$

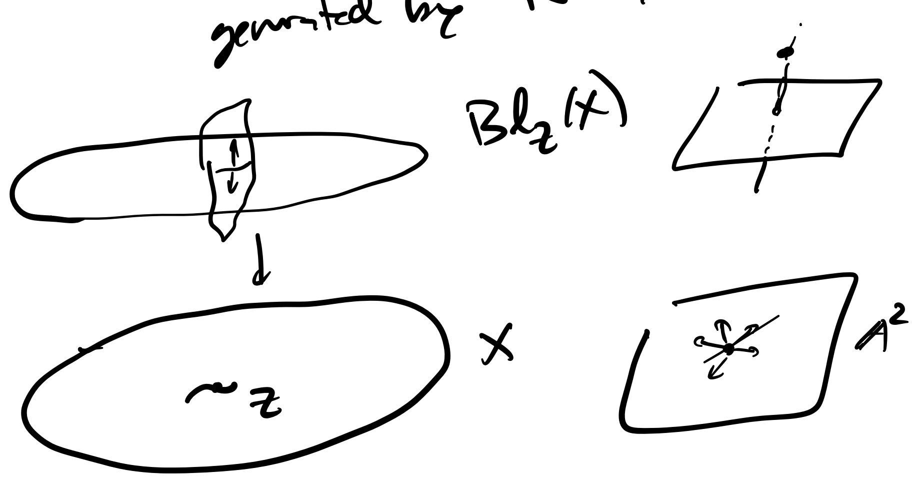
Next most important example: Blow up.

if $X = \operatorname{Spec} R$, $Z \subset X$ closed w/ ideal $I \triangleleft R$

$Bl_Z X = \operatorname{Proj} R \oplus It \oplus I^2 t \oplus \cdots$

i.e. the subring of $R[t]$ (graded in t)

generated by R & It



Relative proj

Sheaves of graded $\mathcal{O}_X$-algebras

Proj. morphisms

Def A proj morphism is a morphism $X \xrightarrow{\pi} Y$ s.t.

$\exists$ cover $\{U_i \to Y\}$ s.t. $\pi^{-1}(U_i) \to U_i = \operatorname{Spec} A_i$

$U_i = \operatorname{Spec} A_i$

s.t. $\exists$ S_i graded A_i-algebras w/ $(S_i)_0 = A_i$

and isos $\pi^{-1}(U_i) \simeq \operatorname{Proj} S_i$.

$$\begin{tikzcd} \operatorname{Proj} S_i \simeq \pi^{-1}(U_i) \arrow[r] \arrow[d] & X \arrow[d] & \mathbb{P}^n \times \mathbb{A}^1 \arrow[d] \\ \operatorname{Spec} A_i = U_i \arrow[r] & Y & pt. \end{tikzcd}$$

ex: $Y = \operatorname{Spec} \mathbb{C}$ $\quad X = \operatorname{Proj} \mathbb{C}[t][x_0, \dots, x_n]$ grade by x_i's.

$$= \begin{array}{c} \mathbb{P}^n \times \mathbb{A}^1 \\ \downarrow \\ \mathbb{A}^1 \end{array}$$

Compare

$$\begin{tikzcd} \text{Affine schemes} & \text{Comm. Rng} \arrow[l, "\mathrm{Spec}"'] \arrow[from=1-1, to=1-2, bend right, "\Gamma"'] \end{tikzcd}$$

$$\begin{tikzcd} \text{Proj schemes} & \text{Graded Rngs} \arrow[l, "\mathrm{Proj}"'] \arrow[from=1-1, to=1-2, bend right, "\Gamma_+"'] \end{tikzcd}$$

+ v. ample line bundle.

Exercise: $S = k[x_0, \ldots, x_n]$ $\quad S' = \bigoplus S'_n$

$S'_n \equiv S_{2n}$

$\operatorname{Proj} S \cong \operatorname{Proj} S'$

"Veronese"