Coherent & quasi-coherent sheaves

Def if R a comm. ring, M an R-module, a presentation of an R-module M is a right exact seq:

$$R^I \to R^J \to M \to 0 \quad \text{some } I, J \text{ possibly infinite.}$$

Def M is finitely generated if $\exists\ R^n \twoheadrightarrow M$, n finite

Def M is finitely presented if $\exists$ presentation w/ I, J finite.

Rem: if R is Noeth then M f.g. $\Leftrightarrow$ M Noeth $\Leftrightarrow$ M f.p.

Def A scheme X is loc. Noeth if $\exists$ cover $\{U_i \to X\}$ w/ $U_i = \operatorname{Spec} A_i$, A_i Noeth rgs.

Def $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules (on some ringed space) a (global) presentation of $\mathcal{M}$ is a R. ex. seq. of $\mathcal{O}_X$-mods

$$\mathcal{O}_X^I \to \mathcal{O}_X^J \to \mathcal{M} \to 0.$$

$$1 \in \Gamma(\mathcal{O}_X)$$

Aside: What is a map $\mathcal{O}_X \to \mathcal{M}$?

need $\Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{M})$

$1 \longmapsto m \in \Gamma(\mathcal{M})$

on U, $r \in \mathcal{O}_X(U)$

$r = r \cdot 1 = r \cdot 1|_U \to r \cdot m|_U$

given $m \in \Gamma(\mathcal{M})$ define

$\mathcal{O}_X \longrightarrow \mathcal{M}$ via

$r \in \mathcal{O}_X(U) \longmapsto r \cdot m|_U$

$$\mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\mathcal{O}_X, \mathcal{M}) = \Gamma(\mathcal{M})$$

Def $\mathcal{M}$ is f.g. (globally) if $\mathcal{O}_X^n \twoheadrightarrow \mathcal{M}$ (very strong)

Def $\mathcal{M}$ is finite type if $\mathcal{M}$ is locally finitely generated

i.e. $\exists$ cover $\{U_i \to X\}$ s.t. $\mathcal{M}|_{U_i}$ is fin. generated

i.e. $\mathcal{O}_{U_i}^{n_i} \twoheadrightarrow \mathcal{M}|_{U_i}$ all i.

Def $\mathcal{M}$ is loc. presentable if it locally has a presentation.

Warning: $U \subset X$ open affine $U = \operatorname{Spec} A$ $\mathcal{M}$ shf of $\mathcal{O}_X$-mod

$M = \mathcal{M}(U)$ is an $\mathcal{O}_X(U) = A$-module.

$$\mathcal{O}_X(U)^I \to \mathcal{O}_X(U)^J \twoheadrightarrow \mathcal{M}(U)$$

$$A^I \to A^J \to M \to 0$$

gives me: $\mathcal{O}_U^I \to \mathcal{O}_U^J \to \mathcal{M}|_U \to 0$, need not be exact.

<u>Def</u> $\mathcal{M}$ is <u>loc. finitely presented</u> (lfp) if it is.

<u>Ref</u> If X is a scheme, we'll say that $\mathcal{M}$ is <u>q.coherent</u> if it is loc. presentable.

<u>Ex</u>: if $U \xrightarrow{i} X$ open inclusion. X scheme.

$\mathcal{F}$ a sheaf (of modules) on U, define $i_!\mathcal{F}$ as (extension by 0)

$$i_!\mathcal{F}(V) = \begin{cases} 0 & \text{if } V \not\subset U \\ \mathcal{F}(V) & \text{if } V \subset U \end{cases}$$

but if $U \neq X$, $\Gamma(i_!\mathcal{F}) = 0$

but $(i_! \mathcal{F})_p = \mathcal{F}_p \quad p \in \mathcal{U}$

$(i_! \mathcal{F})_Q = 0 \quad Q \notin \mathcal{U}.$

Coherence: X ringed space

Def $\mathcal{M}$ is coherent if 1) $\mathcal{M}$ is loc. fin. presented & 2) $\forall\, U \subset X$ open, $\mathcal{O}_U^n \xrightarrow{f} \mathcal{M}|_U$ any morphism, then $\ker f$ is loc. fin. gen. (f. type)

Rem: if X is loc. Noeth scheme, 2) always holds.

i.e. coherent $\Leftrightarrow$ 1) loc. fin. pres $\Leftrightarrow$ l. finitely gen.

Def A scheme X is Noetherian if X is loc. Noeth, & q. compact as a top. space.

Theorem if $X = \operatorname{Spec} A$ then $\mathcal{M}$ is q. coh. on X if and only if $\mathcal{M} \simeq \widetilde{M}$ for some A-mod M.

Recall: $\underline{A\text{-mod}} \longrightarrow \underline{\mathcal{O}_X\text{-mod}} \qquad X = \operatorname{Spec} A$

$M \longmapsto \widetilde{M}$

$\tilde{M}(D_f) = M_f = M \otimes_R R_f$

why is this a shef?

check shef prop for cov ... $\mathcal{B}$-shef

$\mathcal{B}$ = basic opens.

$\tilde{M}|_{D_f} = \widetilde{M_f}$ so enough to check on covs of X

suppose $\{D_{f_i} \longrightarrow X\}$ cov i.e. $(f_i) = R$

want to show $M \longrightarrow \prod M_{f_i} \rightrightarrows \prod M_{f_i f_j}$

equalizer.

check $M \hookrightarrow \prod M_{f_i}$

$m \longmapsto 0$ means $f_i^{n_i} m = 0$

$\Rightarrow f_i^N m = 0 \quad N = \max$

$(f_i^N) = R \Rightarrow$

$\sum f_i^N r_i = 1$

$0 = \sum f_i^N r_i m = m$

glue: of $m_i \in M_{f_i}$ s.t

$\underbrace{m_i|_{D_{f_i f_j}} = m_j|_{D_{f_i f_j}}}$

$$m_i = \frac{m_j'}{f_i^{n_i}} = \frac{m_i'}{f_i^M} \qquad (f_i f_j)^N f_j^M m_i' = (f_i f_j)^N f_i^M m_j'$$

$$f_j^{M+N}(f_i^N m_i') = f_i^{M+N}(f_j^N m_j')$$

write: $\sum n_i f_i^{M+N} = 1$

Pretend $\exists m$, $m = \frac{m_i'}{f_i^M}$

$$\sum n_i f_i^{M+N} m = m$$

$$= \sum n_i f_i^{M+N} \frac{m_i'}{f_i^M}$$

$$m = \sum n_i f_i^N m_i'$$

So set $m = \sum n_i f_i^N m_i'$

in R_{f_j}, $\sum n_i f_i^{M+N}(f_j^N m_j') / f_j^{M+N}$

$$f_j^{M+N}(f_i^N m_i') = f_i^{M+N}(f_j^N m_j')$$

$$= \sum_i n_i f_i^{M+N} \frac{m_j'}{f_j^M} = \frac{m_j'}{f_j^M} \underbrace{\sum_i n_i f_i^{M+N}}_{1}$$

$$= \frac{m_j^!}{f_j^! M} = m_j \quad \square.$$

<u>Moral content of thm:</u> is that

$$R^I \to R^J \to M \to 0 \quad \text{exact}$$

<u>does give</u> $\mathcal{O}_X^I \to \mathcal{O}_X^J \to \tilde{M} \to 0$ exact.

<u>Rem:</u> $\tilde{R} = \mathcal{O}_X$

In particular q.coh $\Longleftrightarrow$ locally of form $\tilde{M}$.

<u>Relative Spec</u>

If X a scheme, get a stack on X of sheaves of q.coh $\mathcal{O}_X$-algebras

$X \qquad U \subset X \rightsquigarrow \mathcal{O}_U\text{-alg}$

$$U \longmapsto \mathcal{O}_U\text{-alg}$$

<u>stack</u>: to construct an object $A \in \mathcal{O}_U$-alg.

is equiv. to constructing $A_i \in \mathcal{O}_{U_i}$-alg.

& isoms $\varphi_{ij} : A_i|_{U_i \cap U_j} \to A_j|_{U_i \cap U_j}$ s.t.

stack condition

$$\varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{\cdots} = \varphi_{ik}|_{U_{ijk}}$$

$$\underset{\sim}{\mathcal{O}_X\text{-alg}}(U) \xrightarrow[\text{eq.}]{\sim} \mathrm{Desc}(\{U_i\}, \underset{\sim}{\mathcal{O}_X\text{-alg}})$$

<u>Another stack</u>: Relative schemes

$\underset{\sim}{Sch_X}(U)$ = Cat of U-schemes.

$\underset{\sim}{Aff_X} \subset \underset{\sim}{Sch_X}$ $\quad \underset{\sim}{Aff_X}(U)$ = Cat of Affine U schemes

$Y \to U$ affine if inv. image of affine is affine.

<u>Notice</u>: $\underset{\sim}{Aff_X}(\mathrm{Spec}\, A)$ $\quad \mathrm{Spec}\, A \to X$

$\|$

morphisms $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$

A-algebras B

$\underset{\sim}{Aff_X}(\mathrm{Spec}\, A)$ = q. coherent sheaves of $\mathcal{O}_{\mathrm{Spec}\, A}$-algebras.

$\underset{\sim}{Aff}_X(\operatorname{Spec} A) = \underset{\sim}{\mathcal{O}_X\text{-alg}}(\operatorname{Spec} A)^{op}$

__Remark:__ $\underset{\sim}{Aff}_X$ & $\underset{\sim}{\mathcal{O}_X\text{-alg}}$ stacks,

$(\underset{\sim}{\mathcal{O}_X\text{-alg}})^{op}$ stack locally equivalent.

$\Rightarrow \underset{\sim}{Aff}_X = (\underset{\sim}{\mathcal{O}_X\text{-alg}})^{op}$

__Def:__ $\underset{\sim}{Aff}_X(U)$ = Cat of U-schemes.

ob: $Y \to U$

$U \to X$ __arb. morphism__

mor:
$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & U & \end{tikzcd}$$

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

$\underset{\sim}{Aff}_X(f)$: __$V$-sch__ $\to$ __U-sch__

$$\begin{tikzcd} Y \arrow[d] \arrow[r] & Y \times_V U \arrow[d] \\ V & U \end{tikzcd}$$

$\{U_i \to U\}$ of X-schemes

$$\underset{\sim}{Aff}_X(X) \xleftarrow[\sim]{\text{eq: } \underline{\operatorname{Spec}}} \underset{\sim}{\mathcal{O}_X\text{-alg}}(X)^{op}$$

Cat of affine morphisms $\begin{matrix} Y \\ \downarrow \\ X \end{matrix}$

(Cat of q.coh $\mathcal{O}_X$-alg$)^{op}$

$\mathcal{A}/\mathcal{O}_X$ cover $\{U_i \to X\}$ $U_i = \operatorname{Spec} A_i$

$$\mathcal{A}_i|_{U_i \cap U_j} \xrightarrow{\varphi_{ij}} \mathcal{A}_j|_{U_i \cap U_j}$$

$$\operatorname{Spec} \mathcal{A}_i|_{U_i \cap U_j} \xrightarrow{(\operatorname{Spec} \varphi_{ij})^{-1}} \operatorname{Spec} \mathcal{A}_j|_{U_i \cap U_j}$$

glue $\operatorname{Spec} \mathcal{A}_i$'s to get a rel. scheme

$$\begin{tikzcd}
\operatorname{Spec} A_i \arrow[r] \arrow[d] & \underline{\operatorname{Spec}}\, \mathcal{A} \arrow[d] \\
U_i \arrow[r, hook] & X
\end{tikzcd}$$

stack

$\underset{\sim\sim}{\text{qc } \mathcal{O}_X\text{-alg}}\,(U) = \mathcal{O}_U\text{-alg's}.$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr] & U \arrow[d] \\
& X
\end{tikzcd}$$

cat.

$$\underline{\mathcal{O}_U\text{-alg}} \longrightarrow \underline{\mathcal{O}_V\text{-alg}}$$

$$\mathcal{A} \longrightarrow f^*\mathcal{A}.$$

<u>Def</u> if $X \xrightarrow{f} Y$ schemes, $\mathcal{M}$ a q-coh. $\mathcal{O}_Y$-module then we define $f^*\mathcal{M} = f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

$$f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$$

$$f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$