Cleaning up from end of last time

Recall: let X be a top space, $\mathcal{F}, \mathcal{G}$ sheaves on X (of sets)

$f: \mathcal{F} \to \mathcal{G}$ then TFAE:

1) for each $p \in X$, $f_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective
2) $\forall\, U \subset X$ open, $s \in \mathcal{G}(U)$, $\exists\, \{U_i \to U\}$ cover and $t_i \in \mathcal{F}(U_i)$ s.t. $f(U_i)(t_i) = s|_{U_i}$

$2 \Rightarrow 1$ if $s \in \mathcal{G}(U)$, $\forall\, p \in U$ $\exists\, t_p \in \mathcal{F}_p$ s.t.

the stalk of s, s_p is the image of t_p

if $V \ni p$ open s.t. $t_p = [\tilde{t}_p]$ $\tilde{t}_p \in \mathcal{F}(V)$

then $f(V)(\tilde{t}_p) \in \mathcal{G}(V)$ represents the class s_p

same class as $[s|_V]$ so $\exists\, W_p \ni p$ smaller nbhd

$W_p \subset V$ s.t. $f(V)(\tilde{t}_p)|_{W_p} = s|_{W_p}$

let $t_p' = \tilde{t}_p|_{W_p}$

$$f(W_p)(t_p') = f(W_p)(\tilde{t}_p|_{W_p}) = f(V)(\tilde{t}_p)|_{W_p} = s|_{W_p}$$

done since W_p's cover.

Def If $\mathcal{X}$ is a site, $\mathcal{F}, \mathcal{G}$ sheaves on $\mathcal{X}$,
we say $f: \mathcal{F} \to \mathcal{G}$ is surjective if
$\forall\, U \in ob(\mathcal{X}),\ s \in \mathcal{G}(U)$
$\exists\ \{U_i \to U\}$ cov (in $\mathcal{X}$), and $t_i \in \mathcal{F}(U_i)$
s.t. $f(U_i)(t_i) = s|_{U_i}$

Recall from last time:
$\mathcal{X} \longleftrightarrow$ Cat of schemes/S w/ Zariski topology
we said $\varphi : X \longrightarrow Y$ $\quad (X, Y \in \underline{Shv}(\underline{Sch}/S))$
is representable if $\forall\ Z \in \underline{Sch}/S,\ g: Z \to Y$,
the fiber product $X \times_Y Z\ (= X \times_Y h_Z)$ is representable
i.e. $X \times_Y h_Z \underset{\text{nat iso}}{\simeq} h_W$ some scheme W.

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow \varphi \\ Z & \xrightarrow{g} & Y \end{array}$$

also a scheme $\to X \times_Y Z$

Z scheme

$$\boxed{\begin{array}{c} g: Z \to Y \\ \updownarrow \\ g: h_Z \to Y \\ \updownarrow \text{Yoneda} \\ Y(Z) \end{array}}$$

Similarly, $\varphi : X \to \mathcal{Y}$ is <u>open</u> if it is representable, and $\forall\, Z \to \mathcal{Y}$, Z scheme, the map

$$\underset{\text{scheme}}{X \times_{\mathcal{Y}} Z} \longrightarrow Z \quad \text{is an open immersion}$$

<u>of schemes</u>.

<u>ex:</u> If $X, \mathcal{Y}$ schemes, what does it mean to say $\varphi : h_X \to h_Y$ is surjective?

By def, $\forall\, U \to \mathcal{Y}$, $\exists$ cov $\{U_i \to U\}$ and $U_i \to X$ s.t.

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & X \arrow[d] \\ U \arrow[r] & \mathcal{Y} \end{tikzcd} \qquad \Rightarrow \text{ if } y \in \mathcal{Y} \qquad \begin{tikzcd} \{y_i\} \arrow[r] \arrow[d] & X \arrow[d] \\ y \arrow[r] & \mathcal{Y} \ni y \end{tikzcd}$$

<u>ex:</u> $\operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k[y] \ni f(y)$

$$k[x] \longleftarrow k[y]$$
$$x^2 \longleftarrow y$$

$g \mid f(x^2)$
irred factr.

$$k[x] \longrightarrow k[y]$$
$$x \longmapsto h(y)$$
$$h(x^2) \longleftarrow$$

__ex:__ if $\{V_i \longrightarrow V\}$ open covering then

$$\coprod V_i \longrightarrow V \quad \underline{\text{is}} \text{ Zariski surjective.}$$

__Why?__ if $U \xrightarrow{f} V$ WTS $\exists$ cov $\{U_j \to U\}$

$$U_j \longrightarrow \coprod V_i \quad \text{s.t.} \quad \begin{tikzcd} U_j \arrow[r] \arrow[d] & \coprod V_i \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

consider the cov $U_i = f^{-1}(V_i)$

$$\begin{tikzcd} f^{-1}(V_i) \arrow[r, "f"] \arrow[d] & \coprod V_i \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd}$$

__Def__ If $\mathcal{F}$ is a Zariski sheaf and U_i schemes we say $\{U_i \xrightarrow{f_i} \mathcal{F}\}$ is an open cov if each f_i is representable open, and $\coprod U_i \longrightarrow \mathcal{F}$ surjective as sheaves.

$$\overbrace{V \longrightarrow \mathcal{F}}^{\mathcal{F}(V)} \Rightarrow \exists \text{ cov } \{V_j \to V\} \quad h_{\coprod(U_i)}(V_j)$$

$$\text{s.t.} \quad \begin{tikzcd} V_j \arrow[r] \arrow[d] & \coprod U_i \arrow[d] \\ V \arrow[r] & \mathcal{F} \end{tikzcd}$$

$\coprod U_i \xrightarrow{\psi} \mathcal{F}$ surj $\Longleftrightarrow$ $h_{\coprod U_i} \longrightarrow \mathcal{F}$ surj (sheaves)

$\forall\, s \in \mathcal{F}(V)$ $\exists$ $\{V_j \to V\}$ cov, $t_j \in h_{\coprod(U_i)}(V_j)$

s.t. $\psi(V_j)(t_j) = s|_{V_j}$

renotate: $\boxed{s \in \mathcal{F}(V)} \longleftrightarrow \boxed{h_V \xrightarrow{s} \mathcal{F}} \longleftrightarrow \boxed{V \xrightarrow{s} \mathcal{F}}$

$\boxed{t_j \in h_{\coprod(U_i)}(V_j)} \longleftrightarrow \boxed{h_{V_j} \to h_{\coprod U_i}} \longleftrightarrow \boxed{V_j \to \coprod U_i}$

$$\begin{tikzcd}
V_j \arrow[r, "t_j"] \arrow[d, hook] & \coprod U_i \arrow[d, "\psi"] \\
V \arrow[r, "s"'] & \mathcal{F}
\end{tikzcd}$$

$$\begin{tikzcd}
V_j \arrow[r, "t_j"] \arrow[d, equal] & \coprod U_i \\
\coprod_i V_{ij} &
\end{tikzcd}$$

$V_{ij} = t_j^{-1}(U_i)$

$$\begin{tikzcd}
\coprod_i (W_i) = \coprod_j \coprod_i V_{ij} \arrow[r, "\coprod_j \coprod_i t_j|_{V_{ij}}"] \arrow[d, hook] & \coprod U_i \arrow[d] \\
V \arrow[r] & \mathcal{F}
\end{tikzcd}$$

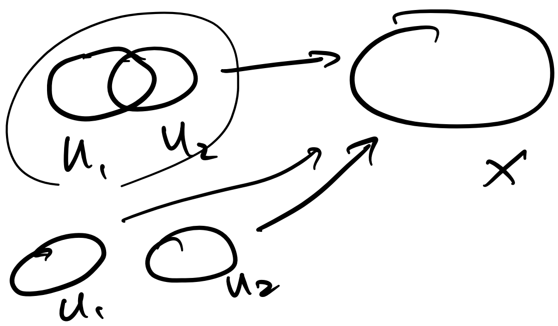
$W_i = \coprod_j V_{ij}$

$$\begin{tikzcd}
\coprod_i \left(\bigcup V_{ij}\right) \arrow[drr] \arrow[ddr, bend right] & & \\
& \coprod_i W_i \arrow[r] \arrow[d] & \coprod U_i \arrow[d] \\
& V \arrow[r] & \mathcal{F}
\end{tikzcd}$$

$W_i \longrightarrow U_i$

$\{W_i \hookrightarrow V\}$ open cov.

$W_i = \coprod V_{ij}$
open in V



i.e. if $V \xrightarrow{g} \mathcal{F}$ V scheme

then

$$\begin{tikzcd}
U_i \times_{\mathcal{F}} V \arrow[r] \arrow[d, "\text{open immersions of schemes}"'] & U_i \arrow[d, "f_i"] \\
V \arrow[r, "g"'] & \mathcal{F}
\end{tikzcd}$$

$$\begin{tikzcd}
U_i \times_{\mathcal{F}} U_j \arrow[r] \arrow[d, "\text{open}"'] & U_i \arrow[d] \\
U_j \arrow[r] & \mathcal{F}
\end{tikzcd}$$

$$\begin{tikzcd}
U_i & & U_j \\
& U_{ij} \arrow[ul, hook, "\text{open}"] \arrow[ur, hook', "\text{open}"'] &
\end{tikzcd}$$

U_{ij}
$\parallel$
$U_i \times_{\mathcal{F}} U_j$

$\Rightarrow \mathcal{F}$ is representable by the scheme obtained by gluing U_i's along

$U_{ij} = U_i \times_{\mathcal{F}} U_j$

Def A Zariski space$/_S$ is a sheaf which admits an open covering by schemes.

$$\coprod \operatorname{Spec} A_{ij}^k \rightrightarrows \coprod \operatorname{Spec} A_i \longrightarrow \mathcal{F}$$

covering of $\operatorname{Spec} A_i \times_{\mathcal{F}} \operatorname{Spec} A_j$

$$\prod A_i \rightrightarrows \prod A_{ij}^k$$

$$A \leftleftarrows B$$

$$X = U_1 \cup U_2$$

$$U_1 \cap U_2 \rightrightarrows U_1 \amalg U_2 \longrightarrow X$$

$$U \underset{\sigma}{\overset{id}{\rightrightarrows}} U \longrightarrow \text{"} U/\sigma \text{"}$$

$U \circlearrowleft C_2$

$$\cdots X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0$$

$$\coprod U_{ijk} \qquad \coprod U_{ij} \qquad \coprod U_i$$

$$U_{ii} \longleftarrow U_i$$

simplicial scheme

$\updownarrow$

"higher stack"

cosimplicial gps

- Sheaves of $\mathcal{O}_X$-modules, q. coh & coherent
- Sheaves of $\mathcal{O}_X$-algebras, relative $\underline{\text{spec}}$.
- Proj, relative proj

Prop. of morphisms: flat, smooth, étale, unramified...
differentials

If $(X, \mathcal{O}_X)$ is ringed space

then a sheaf of $\mathcal{O}_X$ modules on X is a sheaf of Ab. groups $\mathcal{M}$ s.t. $\forall\, U \subset X$ open, $\mathcal{M}(U)$ an $\mathcal{O}_X(U)$-module s.t.

$m \in \mathcal{M}(U)$, $f \in \mathcal{O}_X(U)$ $\quad V \subset U$

$$(f \cdot m)|_V = f|_V \cdot m|_V$$

Category of $\mathcal{O}_X$-modules – generally has great properties.

nice Ab. category, enough injectives (typically)

$(X, \mathcal{O}_X)$ a scheme – enough injectives

if $(X, \mathcal{O}_X)$ $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

then A-modules $\rightsquigarrow$ $\mathcal{O}_X$ modules

$$\underline{A\text{-mod}} \xrightarrow{\sim} \underline{\mathcal{O}_X\text{-mod}}$$

$$M \longrightarrow \widetilde{M}$$

$f \in A$

$\widetilde{M}(D_f) = M \otimes_A A_f$ $\qquad \mathcal{O}_X(D_f) = A_f$

Such $\mathcal{O}_X$-modules $\tilde{M}$ are called quasicoherent.

<u>Prop</u> $\sim$ is fully faithful.

But ~~wrong~~ $\mathcal{O}_X$-mod is much bigger than the subcat $QC\,\mathcal{O}_X$-mod !

<u>Def</u> $\mathcal{M}$ an $\mathcal{O}_X$-mod is q.coh if $\mathcal{M}|_{\operatorname{Spec} A}$ is q.c. all $\operatorname{Spec} A \subset X$ affine open.